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Charges in AdS Spacetimes

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Abstract

We give a review of our recent work on conserved charges in cosmological spacetimes. We compute the mass and the angular momenta of various solutions in D dimensions. This is an extended version of a talk given by B. Tekin. The original material can be found in the related papers in our references.

1. Introduction

Conserved quantities, such as energy-momentum, electric charge, angular momentum, baryon number etc., are important in the description of physical phenomena. In the presence of gravity, definition of certain conserved charges (such as energy) become rather tricky. In the literature, one can find many different definitions which give different results for the same spacetimes. Especially when long-range scalar fields that modify the asymptotic form of the metric are present, one has to be very careful about writing down an energy expression. Our task in this paper is modest: we shall give a review of the techniques of defining conserved charges in asymptotically (Anti)de-Sitter spaces developed by Abbott-Deser (AD) [1] and Deser-Tekin (DT) [2]. [We would like to stress that we do not present new material but simply quote our earlier work and carry out the computations in a little more detail.] These methods are in the same spirit as the Arnowitt-Deser-Misner (ADM) [3] methods which use the Killing symmetries and work for asymptotically flat geometries. For related information, please see [4, 5, 6].

In addition to the cosmological Einstein theory, we will also define the global charges primarily in D dimensional quadratic theories that frequently appear in various low energy string theory or supergravity models. We will first present a reformulation of the original definition of conserved charges in cosmological Einstein theory; then we will derive the generic form of the energy for quadratic gravity theories in D dimensions and specifically study the ghost-free low energy string-inspired model: Gauss-Bonnet (GB) plus Einstein terms [2].

Let us recall that a definition of gauge invariant conserved (global) charges in a diffeomorphism-invariant theory rests on the "Gauss law" and the presence of asymptotic Killing symmetries. More explicitly, in any diffeomorphism-invariant gravity theory, a vacuum satisfying the classical equations of motion is chosen as the background relative to which excitations and any background gauge-invariant properties are defined. Two important model-independent features of these charges are: First, the vacuum itself has zero charge; secondly, they are expressible as surface integrals [1, 2, 3, 7]. [Just to give an example of how various charge definitions in the literature give different expressions, we simply note that in certain prescriptions, different than ours, the background AdS space have non-vanishing energy. For comparison of various charge definitions we refer the reader to a recent paper [8].]

2. Reformulation of Abbott-Deser Charges for generic gravity models

2.1. Conserved Charges

We first look at how conserved charges arise in a generic gravity theory coupled to a covariantly conserved bounded matter source $\tau_{\mu\nu}$. Consider the following equations of motion which either comes from a proper Lagrangian or is endowed with the Bianchi identities and covariant conservation of the matter source:

$$\Phi_{\mu\nu}(g, R, \nabla R, R^2, \ldots) = \kappa\tau_{\mu\nu}, \tag{1}$$

where $\Phi_{\mu\nu}$ is the "Einstein tensor" of a local, invariant, but otherwise arbitrary, gravity action and κ is an effective coupling constant. We work in generic D dimensions.

Now we will decompose our metric into the sum of two parts:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2}$$

where $\bar{g}_{\mu\nu}$ solves (1) for $\tau_{\mu\nu} = 0$ and a deviation part $h_{\mu\nu}$ that vanishes sufficiently rapidly at infinity and is not necessarily small everywhere. [One can also work in the first-order vielbein spin-connection formulation, which is necessary whenever fermionic fields are to be taken into account. Such a computation was carried out recently [6], whose result we shall quote below.]

Separating the field equations (1) into a part linear in $h_{\mu\nu}$ and collecting all other non-linear terms and the matter source $\tau_{\mu\nu}$ in $T_{\mu\nu}$ that constitute the total source, one obtains

$$\mathcal{O}(\bar{g})_{\mu\nu\alpha\beta}h^{\alpha\beta} = \kappa T_{\mu\nu}, \tag{3}$$

$\Phi_{\mu\nu}(\bar{g}, \bar{R}, \bar{\nabla}\bar{R}, \bar{R}^2, \ldots) = 0$, by assumption; the operator $\mathcal{O}(\bar{g})$ depends only on the background metric $\bar{g}_{\mu\nu}$. It is clear that this operator inherits both background Bianchi identity and background gauge invariance namely, $\bar{\nabla}_\mu \mathcal{O}(\bar{g})^{\mu\nu\alpha\beta} = \mathcal{O}(\bar{g})^{\mu\nu\alpha\beta}\bar{\nabla}_\alpha = 0$, from (the Bianchi identities of) the full theory. As a consequence of these invariances, it is guaranteed that if the background $\bar{g}_{\mu\nu}$ admits a set of Killing vectors $\bar{\xi}_\mu^{(a)}$,

$$\bar{\nabla}_\mu \bar{\xi}_\nu^{(a)} + \bar{\nabla}_\nu \bar{\xi}_\mu^{(a)} = 0, \tag{4}$$

then the energy-momentum tensor can be used to construct the following (ordinarily) conserved vector density current :

$$\bar{\nabla}_\mu(\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu^a) \equiv \partial_\mu(\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu^a) = 0. \tag{5}$$

Note that *the* crucial point here is that we are looking for ordinarily conserved charges and we can get this with the help of background Killing vectors. Therefore, up to a constant, the conserved Killing charges are expressed as

$$Q^\mu(\bar{\xi}^a) = \int_{\mathcal{M}} d^{D-1}x\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu^a = \int_\Sigma dS_i \mathcal{F}^{\mu i}. \tag{6}$$

Here $\mathcal{M}$ is a spatial $(D–1)$ hypersurface and Σ is its $(D-2)$ dimensional boundary; $\mathcal{F}^{\mu i}$ is an antisymmetric tensor obtained from $\mathcal{O}(\bar{g})$, whose explicit form, depends on the theory.

After this generic construction let us apply the outlined procedure in the most interesting case: The cosmological Einstein theory. The linearization of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa\tau_{\mu\nu}, \tag{7}$$

follows as

$$R^L_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R} - \frac{1}{2}\bar{g}_{\mu\nu}R^L + \Lambda h_{\mu\nu} + O(h^2) + ... = 0\,,$$

the background terms will be zero because $\bar{g}_{\mu\nu}$ itself satisfies the field equation. Here, $[\nabla_\mu, \nabla_\nu]V_\lambda = R_{\mu\nu\lambda}{}^\sigma V_\sigma$ and $R_{\mu\nu} \equiv R_{\mu\lambda\nu}{}^\lambda$ and the constant curvature vacuum $\bar{g}_{\mu\nu}$ has Riemann, Ricci and scalar curvatures that are

$$\bar{R}_{\mu\lambda\nu\beta} = \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\nu}\bar{g}_{\lambda\beta} - \bar{g}_{\mu\beta}\bar{g}_{\lambda\nu})\,; \tag{8}$$

$$\bar{R}_{\mu\nu} = \bar{R}_{\mu\lambda\nu\beta}\bar{g}^{\beta\lambda} = \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\,; \tag{9}$$

$$\bar{R} = \bar{R}_{\mu\nu}\bar{g}^{\mu\nu} = \frac{2\Lambda}{(D-2)}D\,. \tag{10}$$

We define all terms of second and higher order in $h_{\mu\nu}$ and the matter source $\tau_{\mu\nu}$ to be the gravitational energy-momentum tensor and write $\mathcal{G}^L_{\mu\nu}$ as

$$\mathcal{G}^L_{\mu\nu} \equiv R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{(D-2)}h_{\mu\nu} \equiv \kappa T_{\mu\nu}\,. \tag{11}$$

As can be explicitly checked, the left hand side of obeys the background Bianchi identity

$$\bar{\nabla}_\mu(R^{\mu\nu}_L - \frac{1}{2}\bar{g}^{\mu\nu}R_L - \frac{2\Lambda}{(D-2)}h^{\mu\nu}) = \bar{\nabla}_\mu\mathcal{G}^{\mu\nu}_L = 0 \tag{12}$$

and thus

$$\bar{\nabla}_\mu T^{\mu\nu} = 0\,. \tag{13}$$

In order to write the spatial volume integrals as surface integrals, we need to carry out the linearization of the relevant tensors. In this part that is what we shall do.

We will take the signature to be $(-,+,+,+,...)$. We know that any invertable metric must satisfy $g_{\mu\nu}g^{\nu\alpha} = \delta_\mu{}^\alpha$ Renaming $\delta g_{\mu\nu} = h_{\mu\nu}$,

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,.$$

We linearize the connections $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})$ with the use of $g^{\mu\nu} = \bar{g}^{\mu\nu} - \delta g^{\mu\nu}$

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\nu}(\bar{\nabla}_\alpha\delta g_{\beta\nu} + \bar{\nabla}_\beta\delta g_{\alpha\nu} - \bar{\nabla}_\nu\delta g_{\alpha\beta})\,. \tag{14}$$

The Riemann tensor is

$$R^\mu{}_{\alpha\beta\nu} = \partial_\beta\Gamma^\mu_{\alpha\nu} - \partial_\nu\Gamma^\mu_{\alpha\beta} + \Gamma^\sigma_{\alpha\nu}\Gamma^\mu_{\sigma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\nu}\,, \tag{15}$$

yielding;

$$\delta R^\mu{}_{\alpha\beta\nu} = \bar{\nabla}_\beta(\delta\Gamma^\mu_{\alpha\nu}) - \bar{\nabla}_\nu(\delta\Gamma^\mu_{\alpha\beta})\,. \tag{16}$$

$\mu \leftrightarrow \beta$ contraction gives the linear part Ricci tensor (a.k.a the Palatini identity)

$$\delta R_{\mu\nu} = \bar{\nabla}_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \bar{\nabla}_\mu(\delta\Gamma^\alpha_{\alpha\nu})\,.$$

Now let us write this in terms of $\delta g_{\alpha\beta}$ with use of (14) and manipulating the indices we are left with

$$\delta R_{\mu\nu} = \frac{1}{2}\{\bar{\nabla}^\sigma\bar{\nabla}_\mu\delta g_{\nu\sigma} + \bar{\nabla}^\sigma\bar{\nabla}_\nu\delta g_{\mu\sigma} - \bar{\Box}\delta g_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\}\,, \tag{17}$$

where $h = \bar{g}^{\alpha\sigma} h_{\alpha\sigma}$ and $\bar{\nabla}^\sigma \bar{\nabla}_\sigma \equiv \bar{\Box}$ that is the background d'Alembertian operator.

The linear part of the Ricci scalar reads

$$R_L = \delta R \quad = \quad -\bar{\Box} h + \bar{\nabla}^\sigma \bar{\nabla}^\mu h_{\sigma\mu} - \frac{2}{(D-2)} \Lambda h \,. \tag{18}$$

Now, we are ready to write (11) in terms of the deviation part of the metric. That is,

$$\begin{aligned} \mathcal{G}_L^{\mu\nu} &= R_L^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} R_L - \frac{2}{(D-2)} \Lambda h^{\mu\nu} \\ &= \frac{1}{2}(-\bar{\Box} h^{\mu\nu} - \bar{\nabla}^\mu \bar{\nabla}^\nu h + \bar{\nabla}_\sigma \bar{\nabla}^\nu h^{\sigma\mu} + \bar{\nabla}_\sigma \bar{\nabla}^\mu h^{\sigma\nu}) \\ &\quad - \frac{1}{2} \bar{g}^{\mu\nu}(-\bar{\Box} h + \bar{\nabla}_\sigma \bar{\nabla}_\alpha h^{\sigma\alpha} - \frac{2}{(D-2)} \Lambda h) - \frac{2}{(D-2)} \Lambda h^{\mu\nu} \,. \end{aligned} \tag{19}$$

2.2. Converting the volume integrals to surface Integrals

Recall that there are two facets of a proper conserved charge definition: First, identification of the "Gauss law", whose existence is guaranteed by gauge invariance; second, choice of the proper vacuum, possessing sufficient Killing symmetries with respect to which global, background gauge-invariant, generators can be defined; these will always appear as surface integrals in the asymptotic vacuum [7].

In converting the volume integrals to surface integrals, let us the following route which will be convenient in the higher curvature cases. Collect all terms in the covariant derivative to get surface terms.

$$\begin{aligned} 2\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} &= 2\bar{\xi}_\nu R_L^{\mu\nu} - \bar{\xi}_\nu \bar{g}^{\mu\nu} R_L - \frac{4\Lambda}{(D-2)} \bar{\xi}_\nu h^{\mu\nu} \\ &= \bar{\xi}_\nu \{ -\bar{\nabla}_\rho \bar{\nabla}^\rho h^{\mu\nu} - \bar{\nabla}^\mu \bar{\nabla}^\nu h + \bar{\nabla}_\sigma \bar{\nabla}^\nu h^{\sigma\mu} + \bar{\nabla}_\sigma \bar{\nabla}^\mu h^{\sigma\nu} \} \\ &\quad - \bar{\xi}^\mu \{ -\bar{\nabla}_\rho \bar{\nabla}^\rho h + \bar{\nabla}_\sigma \bar{\nabla}_\nu h^{\sigma\nu} - \frac{2\Lambda}{(D-2)} h \} - \frac{4\Lambda}{(D-2)} \bar{\xi}_\nu h^{\mu\nu} \end{aligned} \tag{20}$$

which can be recast into the form

$$\begin{aligned} 2\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} &= -\bar{\xi}_\nu \bar{\nabla}_\rho \bar{\nabla}^\rho h^{\mu\nu} - \bar{\xi}_\rho \bar{\nabla}^\mu \bar{\nabla}^\rho h + \bar{\xi}_\rho \bar{\nabla}_\nu \bar{\nabla}^\rho h^{\nu\mu} + \bar{\xi}_\nu \bar{\nabla}_\rho \bar{\nabla}^\mu h^{\rho\nu} \\ &\quad + \bar{\xi}^\mu \bar{\nabla}_\rho \bar{\nabla}^\rho h - \bar{\xi}^\mu \bar{\nabla}_\rho \bar{\nabla}_\nu h^{\rho\nu} + \frac{2\Lambda}{(D-2)} \bar{\xi}^\mu h - \frac{4\Lambda}{(D-2)} \bar{\xi}_\nu h^{\mu\nu} \,. \end{aligned} \tag{21}$$

To collect all terms, we use the commutator relation of a vector that gives us the Riemann tensor. In the first, fourth, fifth and sixth terms the Killing vectors are taken inside the covariant derivative with extra terms that will come from the derivative of the Killing vectors. In the second and third terms, places of derivatives must change, after that the Killing vectors can be taken inside the derivative with two additional terms, the second comes from the exchange of derivatives. After these calculations we are left with

$$\begin{aligned} 2\bar{\xi}_\nu G_L^{\mu\nu} &= \bar{\nabla}_\rho \{ -\bar{\xi}_\nu \bar{\nabla}^\rho h^{\mu\nu} - \bar{\xi}^\rho \bar{\nabla}^\mu h + \bar{\xi}^\rho \bar{\nabla}_\nu h^{\mu\nu} + \bar{\xi}_\nu \bar{\nabla}^\mu h^{\rho\nu} + \bar{\xi}^\mu \bar{\nabla}^\rho h - \bar{\xi}^\mu \bar{\nabla}_\nu h^{\rho\nu} \} \\ &\quad + \frac{2\Lambda}{(D-2)} h \bar{\xi}^\mu - \frac{2\Lambda}{(D-2)} h^{\nu\mu} \bar{\xi}_\nu + \frac{2\Lambda}{(D-2)(D-1)} (h^{\nu\mu} \bar{\xi}_\nu - h \bar{\xi}^\mu) \\ &\quad + (\bar{\nabla}_\rho \bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) - (\bar{\nabla}_\rho \bar{\xi}^\mu)(\bar{\nabla}^\rho h) + (\bar{\nabla}_\rho \bar{\xi}^\mu)(\bar{\nabla}_\nu h^{\rho\nu}) \,. \end{aligned}$$

We will look at the last three terms closely:

$$(\bar{\nabla}_\rho \bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) = \bar{\nabla}_\rho (h^{\mu\nu} \bar{\nabla}^\rho \bar{\xi}_\nu) - h^{\mu\nu} (\bar{\nabla}_\rho \bar{\nabla}^\rho \bar{\xi}_\nu) \,.$$

Operating on Killing vector equation, that is $\bar{\nabla}_\mu \bar{\xi}_\nu + \bar{\nabla}_\nu \bar{\xi}_\mu = 0$, with $\bar{\nabla}^\mu$, one gets

$$\bar{\nabla}^\mu \bar{\nabla}_\mu \bar{\xi}_\nu + \bar{\nabla}^\mu \bar{\nabla}_\nu \bar{\xi}_\mu = 0 \,.$$

The second term can be written in the commutator form that is

$$\bar{\Box}\bar{\xi}_\nu + [\bar{\nabla}^\mu, \bar{\nabla}_\nu]\bar{\xi}_\mu = 0\,,$$

or simply

$$\bar{\Box}\bar{\xi}_\nu = -\frac{2\Lambda}{(D-2)}\bar{\xi}_\nu\,.$$

Using this relation, we have

$$(\bar{\nabla}_\rho\bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) = \bar{\nabla}_\rho(h^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\nu) + \frac{2\Lambda}{(D-2)}h^{\mu\nu}\bar{g}_{\nu\lambda}\bar{\xi}^\lambda\,,$$

and

$$(\bar{\nabla}_\rho\bar{\xi}^\mu)(\bar{\nabla}^\rho h) = -\bar{\nabla}_\rho(h\bar{\nabla}^\mu\bar{\xi}^\rho) + \frac{2\Lambda}{(D-2)}h\bar{\xi}^\mu\,.$$

Using the property of a Killing vector,

$$(\bar{\nabla}_\rho\bar{\xi}^\mu)(\bar{\nabla}_\nu h^{\rho\nu}) = -\bar{\nabla}_\rho(h^{\rho\nu}\bar{\nabla}^\mu\bar{\xi}_\nu) - \frac{2\Lambda}{(D-2)(D-1)}(\bar{\xi}_\nu h^{\mu\nu} - \bar{\xi}^\mu h)\,.$$

Finally collecting these results, we have

$$\begin{aligned} 2\bar{\xi}_\nu\mathcal{G}_L^{\mu\nu} = \; & \bar{\nabla}_\rho\{-\bar{\xi}_\nu\bar{\nabla}^\rho h^{\mu\nu} - \bar{\xi}^\rho\bar{\nabla}^\mu h + \bar{\xi}^\rho\bar{\nabla}_\nu h^{\mu\nu} + \bar{\xi}_\nu\bar{\nabla}^\mu h^{\rho\nu} + \bar{\xi}^\mu\bar{\nabla}^\rho h \\ & -\bar{\xi}^\mu\bar{\nabla}_\nu h^{\rho\nu} + h^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\nu + h\bar{\nabla}^\mu\bar{\xi}^\rho - h^{\rho\nu}\bar{\nabla}^\mu\bar{\xi}_\nu\}. \end{aligned} \tag{22}$$

Since the charge densities are surface terms, the Killing charges become

$$\begin{aligned} Q^\mu(\bar{\xi}) = \; & \frac{1}{4\Omega_{(D-2)}G_D}\oint_\Sigma dS_i\{-\bar{\xi}_\nu\bar{\nabla}^i h^{\mu\nu} - \bar{\xi}^i\bar{\nabla}^\mu h + \bar{\xi}^i\bar{\nabla}_\nu h^{\mu\nu} + \bar{\xi}_\nu\bar{\nabla}^\mu h^{i\nu} \\ & +\bar{\xi}^\mu\bar{\nabla}^i h - \bar{\xi}^\mu\bar{\nabla}_\nu h^{i\nu} + h^{\mu\nu}\bar{\nabla}^i\bar{\xi}_\nu + h\bar{\nabla}^\mu\bar{\xi}^i - h^{i\nu}\bar{\nabla}^\mu\bar{\xi}_\nu\}\,. \end{aligned} \tag{23}$$

Here $dS_i \equiv \sqrt{-det\bar{g}}\; d\Omega_i$ where i ranges over $(1,2,....,D-2)$; the charge is normalized by dividing with the (D-dimensional) Newton's constant G_D and the solid angle Ω_{D-2}. We would like to emphasize a very important point which is often missed: Eqn. (23) gives the conserved charges for spacetimes that are asymptotically AdS, which was our goal. But the obviously the formula works for $\Lambda = 0$, namely the asymptotically flat, case. As we have made no assumption on the choice of coordinates, our formula is a coordinate independent expression. Therefore, before calculating the conserved charges Q^0, let us check our claim of coordinate or the "gauge" invariance of our definition, and also we should check if it goes to the ADM charges in the limit of an asymptotically flat background (For Cartesian coordinates that is $\bar{\nabla}_j \to \partial_j$) in which case our timelike Killing vector is $\bar{\xi}_\mu = (1, \mathbf{0})$.

First we will look at the gauge-invariance. Under an infinitesimal diffeomorphism, generated by a vector ζ (not to be confused with our Killing vector!), the deviation part of the metric transforms as

$$\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu\zeta_\nu + \bar{\nabla}_\nu\zeta_\mu\,. \tag{24}$$

First we will look at the linear Ricci scalar, we have

$$\delta_\zeta R_L = -\bar{\Box}\delta_\zeta h + \bar{\nabla}^\sigma\bar{\nabla}^\mu\delta_\zeta h_{\sigma\mu} - \frac{4\Lambda}{(D-2)}\bar{\nabla}^\mu\zeta_\mu\,.$$

which yields

$$\begin{aligned} \delta_\zeta R_L &= -\bar{g}^{\mu\nu}\bar{\Box}(\bar{\nabla}_\mu\zeta_\nu + \bar{\nabla}_\nu\zeta_\mu) + \bar{\nabla}^\sigma\bar{\nabla}^\mu(\bar{\nabla}_\sigma\zeta_\mu + \bar{\nabla}_\mu\zeta_\sigma) - \frac{4\Lambda}{(D-2)}\bar{\nabla}^\mu\zeta_\mu \\ &= -2\bar{\Box}\bar{\nabla}^\mu\zeta_\mu + \bar{\nabla}^\sigma\bar{\nabla}^\mu\bar{\nabla}_\sigma\zeta_\mu + \bar{\nabla}^\sigma\bar{\Box}\zeta_\sigma - \frac{4\Lambda}{(D-2)}\bar{\nabla}^\mu\zeta_\mu\,. \end{aligned}$$

Let us look at the second and third terms carefully:

$$[\bar{\nabla}^{\mu}, \bar{\nabla}_{\sigma}]\zeta_{\mu} = \bar{\nabla}^{\mu}\bar{\nabla}_{\sigma}\zeta_{\mu} - \bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}\zeta_{\mu}\,,$$

$$\bar{\nabla}^{\mu}\bar{\nabla}_{\sigma}\zeta_{\mu} = \frac{2\Lambda}{(D-2)}\zeta_{\sigma} + \bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}\zeta_{\mu}\,.$$

Therefore, we have

$$\bar{\nabla}^{\sigma}\bar{\nabla}^{\mu}\bar{\nabla}_{\sigma}\zeta_{\mu} = \frac{2\Lambda}{(D-2)}\bar{\nabla}^{\sigma}\zeta_{\sigma} + \bar{\Box}\bar{\nabla}^{\mu}\zeta_{\mu}\,.$$

In the third term the same calculations can be done, yielding

$$\bar{\nabla}^{\sigma}\bar{\Box}\zeta_{\sigma} = \frac{2\Lambda}{(D-2)}\bar{\nabla}^{\beta}\zeta_{\beta} + \bar{\Box}\bar{\nabla}^{\sigma}\zeta_{\sigma}$$

We have the background gauge invariance of the linear Ricci scalar

$$\delta_{\zeta}R_{L} = 0\,.$$

Therefore

$$\delta_{\zeta}\mathcal{G}^{L}_{\mu\nu} = \delta_{\zeta}R^{L}_{\mu\nu} - \frac{2\Lambda}{(D-2)}\delta_{\zeta}h_{\mu\nu}.$$

We then have

$$\begin{aligned}\delta_{\zeta}\mathcal{G}^{L}_{\mu\nu} \;=\; & \frac{1}{2}(-\bar{\Box}\bar{\nabla}_{\mu}\zeta_{\nu} - \bar{\Box}\bar{\nabla}_{\nu}\zeta_{\mu} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\nabla}^{\beta}\zeta_{\beta} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\nabla}^{\alpha}\zeta_{\alpha} \\ & +\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}\bar{\nabla}_{\sigma}\zeta_{\mu} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}\zeta_{\sigma} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\zeta_{\nu} + \bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\zeta_{\sigma}) \\ & -\frac{2\Lambda}{(D-2)}(\bar{\nabla}_{\mu}\zeta_{\nu} + \bar{\nabla}_{\nu}\zeta_{\mu})\,.\end{aligned}$$

Just as before, let us look at the terms that are in the second line: The fifth term is:

$$\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}\bar{\nabla}_{\sigma}\zeta_{\mu} = \frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\nu\mu}\bar{\nabla}^{\sigma}\zeta_{\sigma} - \bar{g}_{\sigma\mu}\bar{\nabla}^{\sigma}\zeta_{\nu}) + \bar{\nabla}^{\sigma}\bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}\zeta_{\mu}\,.$$

The sixth term is:

$$\bar{\nabla}^{\sigma}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}\zeta_{\sigma} = \frac{2\Lambda}{(D-2)(D-1)}(\bar{\nabla}_{\nu}\zeta_{\mu} - \bar{\nabla}_{\mu}\zeta_{\nu}) + \bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\zeta_{\sigma}\,.$$

The third term is:

$$\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\zeta_{\nu} = \frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\mu\nu}\bar{\nabla}^{\sigma}\zeta_{\sigma} - \bar{g}_{\sigma\nu}\bar{\nabla}^{\sigma}\zeta_{\mu}) + \bar{\nabla}^{\sigma}\bar{\nabla}_{\sigma}\bar{\nabla}_{\mu}\zeta_{\nu}\,.$$

The fourth term is:

$$\begin{aligned}\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\zeta_{\sigma} \;=\; & \frac{2\Lambda}{(D-2)}\bar{\nabla}_{\mu}\zeta_{\nu} + \frac{2\Lambda}{(D-2)(D-1)}(\bar{\nabla}_{\mu}\zeta_{\nu} - \bar{g}_{\mu\nu}\bar{\nabla}^{\sigma}\zeta_{\sigma}) \\ & +\frac{2\Lambda}{(D-2)}\bar{\nabla}_{\nu}\zeta_{\mu} + \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\nabla}^{\sigma}\zeta_{\sigma}\,.\end{aligned}$$

Collecting all these, terms we end up with

$$\delta_{\zeta}\mathcal{G}^{L}_{\mu\nu} = 0\,,$$

which means that $\mathcal{G}^L_{\mu\nu}$ is gauge-invariant. Therefore, we have $\delta_\zeta R^L_{\mu\nu} = \frac{2\Lambda}{(D-2)}\delta_\zeta h_{\mu\nu}$.

Hence $\delta_\zeta Q^\mu = 0$; that is, the Killing charge is indeed background gauge-invariant.

Now we will examine (23) in the limit of an asymptotically flat background, which should yield the ADM charge. Let us just look at the mass to begin with. With $\bar{\xi}_\mu = (1, \mathbf{0})$, we have $\bar{\xi}_i = 0$, $\bar{\xi}_0 = 1$ and $\bar{\xi}^0 = -1$ in flat space with the metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. We have $h = -h_{00} + h_{ii}$. Being in Cartesian coordinates, we can replace the covariant derivatives with the partial derivatives ($\bar{\nabla}_j \to \partial_j$). Hence,

$$M = \frac{1}{4\Omega_{(D-2)}G_D}\oint_\Sigma dS_i\{\bar{\xi}_0\partial^0 h^{i0} - \bar{\xi}_0\partial^i h^{00} - \bar{\xi}_0\partial^i h - \bar{\xi}_0\partial^0 h^{i0} + \partial_j h^{ij}\}\,.$$

Writing h explicitly yields

$$\begin{aligned} M &= \frac{1}{4\Omega_{(D-2)}G_D}\oint_\Sigma dS_i\{-\partial^i h^{00} - \partial^i h_{jj} + \partial^i h_{00} + \partial_j h^{ij}\} \\ &= \frac{1}{4\Omega_{(D-2)}G_D}\oint_\Sigma dS_i\{\partial_j h^{ij} - \partial^i h_{jj}\} \end{aligned}$$

which is the usual ADM mass [3].

Instead of the metric, we could have worked with the vielbein and the spin connection (as is done in theories where there are fermions). Let us simply quote the final result for the expression of the charge in that case:

$$\begin{aligned} Q^a(\bar{\xi}) = \frac{1}{4\,\Omega_{D-2}\,G_D}\int_{\partial M} dS_i\,\{&-\bar{\xi}^a\,\bar{D}^b\,\varphi^i{}_b + \varphi^{bi}\,\bar{D}_b\,\bar{\xi}^a - \varphi^b{}_b\,\bar{D}^i\,\bar{\xi}^a + \bar{\xi}^a\,\bar{D}^i\,\varphi^b{}_b \\ &-\bar{\xi}^i\,\bar{D}^a\,\varphi^b{}_b + \bar{\xi}^i\,\bar{D}^b\,\varphi^a{}_b - \bar{\xi}_b\,\bar{D}^i\,\varphi^{ab} + \varphi^{ab}\,\bar{D}^i\,\bar{\xi}_b + \bar{\xi}_b\,\bar{D}^a\,\varphi^{ib}\}\,. \end{aligned} \tag{25}$$

where the full vielbein is decomposed into a background and a deviation parts.

$$e^a \equiv \bar{e}^a + \varphi^a{}_b\,\bar{e}^b. \tag{26}$$

We choose the background to be AdS. The details are to be found in [6].

3. The Energy of Schwarzschild (Anti)de-Sitter Solutions

We can now evaluate the energy of Schwarzschild-de Sitter (SdS) solutions.

In static coordinates, the line element of D-dimensional SdS reads

$$ds^2 = -\left(1 - (\frac{r_0}{r})^{D-3} - \frac{r^2}{l^2}\right)dt^2 + \left(1 - (\frac{r_0}{r})^{D-3} - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{D-2}\,, \tag{27}$$

where $l^2 \equiv (D-2)(D-1)/2\Lambda > 0$. The background ($r_0 = 0$) Killing vector is $\bar{\xi}^\mu = (-1, \mathbf{0})$, which is timelike everywhere for AdS ($l^2 < 0$), but remains timelike for dS ($l^2 > 0$) only inside the cosmological horizon: $\bar{g}_{\mu\nu}\bar{\xi}^\mu\bar{\xi}^\nu = -(1 - \frac{r^2}{l^2})$ [2].

The $D = 4$ Case To see how far we can go with the de-Sitter case, let us concentrate on $D = 4$ first and calculate the surface integral not at $r \to \infty$, but at some finite distance r from the origin; this will not be gauge-invariant, since energy is to be measured only at infinity. Nevertheless, for dS space (which has a horizon that keeps us from going smoothly to infinity), let us first keep r finite as an intermediate step. The integral becomes

$$E(r) = \frac{r_0}{2G}\frac{(1 - \frac{r^2}{l^2})}{(1 - \frac{r_0}{r} - \frac{r^2}{l^2})}\,. \tag{28}$$

For AdS, take $r \to \infty$ and we get the usual mass $\frac{r_0}{2G}$. For dS, we can only consider small r_0 limit, which do not change the location of the background horizon, that also yields $\frac{r_0}{2G}$ [2].

Energy For D Dimensions In this case $h \approx 0$. From the line element $g_{00}, g_{rr}, \bar{g}_{00}, \bar{g}_{rr}$ and the corresponding $h_{\mu\nu}$ terms can be calculated. Since $r_0 = 0$ for the background. We have

$$h_{00} = (\frac{r_0}{r})^{D-3}, \quad \bar{g}^{00}\bar{g}^{00}h_{00} = h^{00} = \frac{(\frac{r_0}{r})^{D-3}}{(1-\frac{r^2}{l^2})^2}. \tag{29}$$

$$h_{rr} = \frac{(\frac{r_0}{r})^{D-3}}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})(1-\frac{r^2}{l^2})}, \quad h^{rr} = \frac{(\frac{r_0}{r})^{D-3}(1-\frac{r^2}{l^2})}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})}. \tag{30}$$

We have

$$Q^0 = \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{\bar{\xi}_0\bar{\nabla}^0 h^{r0} - \bar{\xi}_0\bar{\nabla}^r h^{00} + h^{00}\bar{\nabla}^r\bar{\xi}_0 - h^{rr}\bar{\nabla}^0\bar{\xi}_r + \bar{\nabla}_\nu h^{r\nu}\}$$

where the constant factors come from our normalization of the charges and the integration element dS_i.

$$Q^0 = \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{\bar{\xi}^0\bar{\nabla}_0 h^{r0} - \bar{\xi}_0\bar{\nabla}^r h^{00} + h^{00}\bar{\nabla}^r\bar{\xi}_0 - h^{rr}\bar{\nabla}^0\bar{\xi}_r + \bar{\nabla}_0 h^{r0} + \bar{\nabla}_i h^{ri}\}.$$

Playing with indices and setting $h = 0$ we get

$$Q^0 = \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{-\partial_r h_{00} + h_{00}\bar{g}^{00}\partial_r\bar{g}_{00} + \partial_r h^{rr} + h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr} + \frac{1}{2}h^{rr}\bar{g}^{ij}\partial_r\bar{g}_{ij}\}.$$

In the limit of $r \to \infty$ we get the energy in D-dimensions as

$$E = \frac{(D-2)}{4G_D} r_0^{D-3}. \tag{31}$$

Here r_0 can be arbitrarily large in the AdS case but must be small in dS [2].

The $D = 3$ Case Let us note that analogous computations can also be carried out in $D = 3$; the proper solution to consider is

$$ds^2 = -(1 - r_0 - \frac{r^2}{l^2})dt^2 + (1 - r_0 - \frac{r^2}{l^2})^{-1}dr^2 + r^2 d\phi^2, \tag{32}$$

for which the energy is $E = r_0/2G$ again, but now r_0 is a dimensionless constant and the Newton constant G has dimensions of $1/mass$ [2, 9].

Adding electric charge will not change the answer in the solutions considered above (except for the $D = 3$ case, which is a little non-trivial.) We next consider the recently found D dimensional Kerr-AdS solutions and compute their masses and angular momenta [10].

4. Conserved Charges of Higher D Kerr-AdS Spacetimes

Let us now calculate the conserved charges of the metrics [11] for $D > 3$. [We shall treat the special $D = 3$ case at the end]. They have the Kerr-Schild form [12, 13]

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}(k_\mu\, dx^\mu)^2, \tag{33}$$

in terms of the de Sitter metric

$$\begin{aligned} d\bar{s}^2 \;\; = \;\; & -W\,(1-\Lambda\, r^2)\, dt^2 + F\, dr^2 + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{1+\Lambda\, a_i^2}\, d\mu_i^2 + \sum_{i=1}^{N}\frac{r^2+a_i^2}{1+\Lambda\, a_i^2}\,\mu_i^2\, d\phi_i^2 \\ & + \frac{\Lambda}{W\,(1-\Lambda\, r^2)}\Big(\sum_{i=1}^{N+\epsilon}\frac{(r^2+a_i^2)\,\mu_i\, d\mu_i}{1+\Lambda\, a_i^2}\Big)^2. \end{aligned} \tag{34}$$

Here $\epsilon = 0/1$ for odd/even, dimensions and $D = 2N + 1 + \epsilon$. The null 1-form reads

$$k_\mu\, dx^\mu = F\, dr + W\, dt - \sum_{i=1}^{N} \frac{a_i\, \mu_i^2}{1 + \Lambda\, a_i^2} d\phi_i\,, \tag{35}$$

with

$$U \equiv r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{N} (r^2 + a_j^2),\; W \equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{1 + \Lambda\, a_i^2},\; F \equiv \frac{1}{1 - \Lambda\, r^2} \sum_{i=1}^{N+\epsilon} \frac{r^2\, \mu_i^2}{r^2 + a_i^2}\,. \tag{36}$$

To find the energy and angular momenta corresponding to (33), we must compute the charges Q^0 for the corresponding Killing vectors: for the energy we shall take $\bar{\xi}^\mu = (-1, \vec{0})$ and each angular momentum has the appropriate unit entry $(0, \ldots 1_i \ldots 0)$. Then

$$Q^0 = \frac{1}{4\Omega_{D-2} G_D} \int_\Sigma dS_r \Big\{ g_{00} \bar{\nabla}^0 h^{r0} + g_{00} \bar{\nabla}^r h^{00} + h^{0\nu} \bar{\nabla}^r \bar{\xi}_\nu - h^{r\nu} \bar{\nabla}^0 \bar{\xi}_\nu + \bar{\nabla}_\nu h^{r\nu} \Big\}\,. \tag{37}$$

Using the energy Killing vector, we obtain

$$\begin{aligned} E_D = \frac{1}{4\Omega_{D-2} G_D} \int_\Sigma dS_r \Big\{ & g_{00} g^{rr} \partial_r h^{00} + \frac{1}{2} h^{00} g^{rr} \partial_r g_{00} - \frac{m}{U} g^{00} \partial_r g_{00} + 2m \partial_r U^{-1} \\ & + \frac{2m}{U} g^{rr} \partial_r g_{rr} - \frac{m}{U} g^{rr} k^i k^j \partial_r g_{ij} + \frac{m}{U} g^{ij} \partial_r g_{ij} \Big\}. \end{aligned} \tag{38}$$

As in the previous section, to compute E_D, one needs the large r behavior of the integrand I of (38); since

$$g_{00} \to W \Lambda r^2, \qquad F \to \frac{-1}{\Lambda r^2}, \qquad U \to r^{D-3}, \qquad k^\phi \to \frac{a_\phi}{r^2}\,, \tag{39}$$

then

$$I = \frac{2m}{r^{D-2}} [(D-1)W - 1]\,. \tag{40}$$

For the sake of completeness, let us also note how the determinant is calculated,

$$\det g = -W(1 - \Lambda r^2) F \prod_{i=1}^{N} \frac{(r^2 + a_i^2)\mu_i^2}{1 + \Lambda a_i^2} \det M\,. \tag{41}$$

Here M is the matrix representing the coefficients of the form $d\mu_i d\mu_j$ in the metric, which can be expressed as (no repeated index summation),

$$M_{ij} = A_i \delta_{ij} + B_i B_j + C_i C_j \tag{42}$$

where

$$\begin{aligned} A_i &= \frac{(r^2 + a_i^2)}{1 + \Lambda a_i^2}, \qquad\qquad B_i = \sqrt{\frac{(r^2 + a_{N+\epsilon}^2)}{1 + \Lambda a_{N+\epsilon}^2}} \frac{\mu_i}{\mu_n} \\ C_i &= \sqrt{\frac{\Lambda}{W(1 - \Lambda r^2)}} \Big(\frac{(r^2 + a_i^2)}{1 + \Lambda a_i^2} - \frac{(r^2 + a_{N+\epsilon}^2)}{1 + \Lambda a_{N+\epsilon}^2} \Big) \mu_i\,. \end{aligned} \tag{43}$$

Then we have

$$\det M = \prod_{i=1}^{N+\epsilon-1} A_i \sum_{i=1}^{N+\epsilon-1} \Big\{ \frac{B_i^2}{A_i} + \frac{C_i^2}{A_i} + \sum_{j \neq i}^{N+\epsilon-1} \frac{B_i^2 C_j^2}{A_i A_j} - \sum_{j \neq i}^{N+\epsilon-1} \frac{B_i B_j C_j C_i}{A_i A_j} \Big\}\,. \tag{44}$$

Inserting (43) in the above equation, one gets

$$\det M = \frac{1}{W\mu_{N+\epsilon}^2}\prod_{i=1}^{N}\frac{1}{1+\Lambda a_i^2}. \tag{45}$$

Using equations (45,41,40) the energy of the D dimensional rotating black hole becomes

$$E_D = \frac{m}{\Xi}\sum_{i=1}^{\frac{D-1-\epsilon}{2}}\left\{\frac{1}{\Xi_i} - (1-\epsilon)(\frac{1}{2})\right\}. \tag{46}$$

where

$$\Xi \equiv \prod_{i=1}^{\frac{D-1-\epsilon}{2}}(1+\Lambda a_i^2), \quad \Xi_i \equiv 1+\Lambda a_i^2. \tag{47}$$

This expression reduces to the standard limits $a_i \to 0$ and $\Lambda \to 0$, and agrees (up to a constant factor) with those of [14, 15].

The computation of angular momenta follows along similar lines. Consider a given, say that i^{th} (which we call the ϕ) component, *i.e.*, the Killing vector $\xi^\mu_{(i)} = (0, ..., 0, 1_i, 0, ..)$. Then the corresponding Killing charge becomes

$$\begin{aligned} Q^0 &= \frac{1}{4\Omega_{D-2}G_D}\int_\Sigma dS_r\left\{g_{\phi\phi}\bar\nabla^0 h^{r\phi} - g_{\phi\phi}\bar\nabla^r h^{0\phi} + h^{0\nu}\bar\nabla^r\bar\xi_\nu - h^{r\nu}\bar\nabla^0\bar\xi_\nu\right\} \\ &= \frac{1}{4\Omega_{D-2}G_D}\int_\Sigma dS_r\left\{-g_{\phi\phi}g^{rr}g^{00}\partial_r h_0^\phi\right\}. \end{aligned} \tag{48}$$

Once again the integrand can be calculated to be

$$I = \frac{(D-1)2ma_i\mu_i^2}{r^{D-2}(1+\Lambda a_i^2)}. \tag{49}$$

Putting the pieces together, the angular momentum is

$$J_i = \frac{ma_i}{\Xi\Xi_i}. \tag{50}$$

This expression again agrees with [14, 15]. Note that, unlike in the energy expression, ϵ does not appear here since even dimensional spaces have as many independent 2-planes as the odd dimensional spaces with one lower dimension. For even dimensions, there is a nice relation between the energy and the angular momentum $E = \sum_i \frac{J_i}{a_i}$.

Having computed the desired conserved charges for Kerr-AdS spacetimes in $D > 3$, let us briefly turn our attention to the $D = 3$ BTZ black hole [16]. This solution has long been studied but we recompute the charges with our method for the sake of completeness. The BTZ black hole differs from its higher dimensional counterparts in one very important aspect: for it, AdS is not the correct-vacuum-background [16]. The full metric is

$$ds^2 = (M - \Lambda r^2)dt^2 + \frac{dr^2}{-M + \Lambda r^2 + \frac{a^2}{4r^2}} - adt\, d\phi + r^2 d\phi^2, \tag{51}$$

The background metric corresponds to $M = 0$ and AdS corresponds to $M = -1$. Only AdS with $J = 0$ is allowed for $M < 0$: the others have naked singularities. So we consider $M > 0$ and compute the charges following our calculations above (about the $M = 0$ background.) We get the usual answers

$$E = M, \qquad J = a. \tag{52}$$

BTZ black holes also solve the more general topologically massive gravity equations, where the Einstein term is augmented by the Cotton tensor [17],

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \kappa \tau_{\mu\nu} . \tag{53}$$

Conserved charges in this model were obtained in [4], in terms of those of the Einstein model Q^{μ}_{E},

$$\begin{aligned} Q^{\mu}(\bar{\xi}) &= Q^{\mu}_{E}(\bar{\xi}) + \frac{1}{2\mu} \oint dS_i \left\{ \epsilon^{\mu i \beta} \mathcal{G}^{L}{}_{\nu\beta} \bar{\xi}^{\nu} + \epsilon^{\nu i}{}_{\beta} \mathcal{G}_{L}{}^{\mu\beta} \bar{\xi}_{\nu} + \epsilon^{\mu\nu\beta} \mathcal{G}^{L\, i}{}_{\beta} \bar{\xi}_{\nu} \right\} \\ &+ \frac{1}{2\mu} Q^{\mu}_{E}(\epsilon \bar{\nabla} \bar{\xi}), \end{aligned} \tag{54}$$

where $Q^{\mu}_{E}(\epsilon\bar{\nabla}\bar{\xi})$ is the Einstein form but $\bar{\xi}$ is replaced with its curl. Once the contributions of the Cotton parts are computed the mass and the angular momentum of the BTZ black hole reads:

$$E = M - \frac{\Lambda a}{\mu}, \qquad J = a - \frac{M}{\mu}, \tag{55}$$

a shift in values that may be compared with those for gravitational anyons [18], (linearized) solutions of TMG but not of pure D=3 Einstein.

There are also several other solutions which have topologically non-trivial asymptotics which we shall briefly touch here. Some of these solutions have negative total energy compared to the background spacetimes [6].

The AdS Soliton

Consider the "AdS Soliton" of Horowitz-Myers [19]

$$ds^2 = \frac{r^2}{\ell^2} \left[\left(1 - \frac{r_0^{p+1}}{r^{p+1}} \right) d\tau^2 + \sum_{i=1}^{p-1} (dx^i)^2 - dt^2 \right] + \left(1 - \frac{r_0^{p+1}}{r^{p+1}} \right)^{-1} \frac{\ell^2}{r^2} dr^2, \tag{56}$$

which was obtained by the double analytic continuation of a near extremal p-brane solution. Here x^i $(i = 1, ..., p-1)$ and the t variables denote the coordinates on the "brane" and $r \geq r_0$. To avoid a conical singularity at $r = r_0$, τ necessarily has a period $\beta = 4\pi\ell^2/(r_0(p+1))$. Its energy was computed in [19] using the method of [20]. Here we compute the energy using the method described so far. The background $(r_0 = 0)$ is the usual globally AdS spacetime in the horospherical coordinates, with the timelike Killing vector

$$\bar{\xi}^{\mu} = (-1, 0, ..., 0). \tag{57}$$

Defining the metric perturbation as outlined above and carrying out the integrations, we have

$$E = -\frac{V_{D-3}\, \pi}{(D-1)\, \Omega_{D-2}\, G_D} \frac{r_0^{D-2}}{\ell^{D-2}}, \tag{58}$$

where V_{D-3} is the volume of the compact dimensions. Up to trivial charge normalizations, our result matches that of [19], which uses the energy definition of Hawking-Horowitz [20].

Eguchi-Hanson Solitons

Recently, Clarkson and Mann [21] found very interesting solutions to the *odd* dimensional cosmological (for both signs) Einstein equations. These solutions resemble the even dimensional Eguchi-Hanson metrics [22] - thus the name Eguchi-Hanson solitons - and asymptotically approach AdS/Z_p, where $p \geq 3$. As shown in [21], these solutions have lower energy compared to the global AdS spacetimes (or the global AdS/Z_p spacetimes). The energies of these solutions (for the case of 5 dimensions) were computed in [21] with the help of the boundary counterterm method [23, 24]. It is important to note that boundary counterterm method needs to be worked out for a given fixed dimension. Here, we use the prescription outlined in the previous section and find the energy of the EH solitons for generic odd dimensions. For a detailed description of the metrics, we refer the reader to [21]. We simply quote their result: the EH soliton reads

$$\begin{aligned} ds^2 &= -g(r)\,dt^2 + \left(\frac{2r}{D-1}\right)^2 f(r)\left[d\psi + \sum_{i=1}^{(D-3)/2}\cos\theta_i\,d\phi_i\right]^2 \\ &\quad + \frac{dr^2}{g(r)\,f(r)} + \frac{r^2}{D-1}\sum_{i=1}^{(D-3)/2}\left(d\theta_i^2 + \sin^2\theta_i\,d\phi_i^2\right), \end{aligned} \tag{59}$$

and the metric functions are given by

$$g(r) = 1 \mp \frac{r^2}{\ell^2}, \quad f(r) = 1 - \left(\frac{a}{r}\right)^{D-1}. \tag{60}$$

In the AdS case, to remove the string-like singularity at $r=a$, one finds that ψ has a period $4\pi/p$ and there is a constraint on the parameter a:

$$a^2 = \ell^2\,(\frac{p^2}{4} - 1). \tag{61}$$

The background is obtained simply by setting $a=0$ in (60). The details of the energy (the charge for $\bar{\xi}^\mu = (-1,0,...,0)$) computation is rather lengthy and not particularly illuminating to present here. Instead, we will only write down our result. For convenience, we define

$$E(\bar{\xi}) \equiv \frac{1}{4\,\Omega_{D-2}\,G_D}\int_{\partial M} dS_r\,\mathcal{E}(\bar{\xi}),$$

and only present $\mathcal{E}(\bar{\xi})$, in the $r\to\infty$ limit:

$$\lim_{r\to\infty}\mathcal{E}(\bar{\xi}) = -\frac{2\,a^{D-1}}{\ell^2\,(D-1)^{(D-1)/2}}\prod_{i=1}^{(D-3)/2}\sin\theta_i. \tag{62}$$

After the angular integrations are carried out, one obtains the energy of the EH soliton in generic odd dimensions

$$E = -\frac{(4\pi)^{(D-1)/2}\,a^{D-1}}{p\,\ell^2\,(D-1)^{(D-1)/2}\,\Omega_{D-2}\,G_D}. \tag{63}$$

Specifically, when $D=5$, one finds

$$E = -\frac{a^4}{4\,p\,\ell^2\,G_5}. \tag{64}$$

We note that this result differs from that of [21] in two respects: one of which is a trivial numerical factor that can be attributed to normalization of the conserved charges; the second, and the more important one, is the presence of an additive constant which is exactly equal to the energy of the AdS/Z_p spacetime. Recall that in the formalism we use, the background always has zero energy, unlike the boundary counterterm method for which it has a finite energy.

"Taub-NUT-Reissner-Nordström" solution

We can compute the masses of the new charged solutions [25, 26, 27] in AdS spacetimes that have non-trivial topology. Here, we only consider two examples that were presented in [25]. In $D=4$, the "Taub-NUT-Reissner-Nordström" solution reads

$$ds^2 = -F(r)\,(dt - 2\,N\,\cos\theta\,d\phi)^2 + \frac{dr^2}{F(r)} + (r^2+N^2)\,(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{65}$$

where N is the nut charge and

$$F(r)=\frac{r^4+(\ell^2+6N^2)\,r^2-2\,m\,\ell^2\,r-3N^4+\ell^2\,(q^2-N^2)}{\ell^2\,(r^2+N^2)}. \tag{66}$$

To find the energy of this solution, the correct background (that has zero energy) needs to be carefully chosen. If we naively set $m=q=0$ and the nut charge $N=0$, then the energy of the solution with nonzero m,q,N diverges. This is to be expected since $N=0$ solution is *not* in the same topological class as that of the $N\neq 0$ solutions. The background has to be chosen as $m=q=0$ but $N\neq 0$ as was shown by Deser-Soldate [28] in the case of the (asymptotically locally flat) Kaluza-Klein monopole. In the light of these arguments, one gets

$$E=\frac{m}{G_4}. \tag{67}$$

In $D=6$, the metric, for the details of which we refer to [25], reads

$$\begin{aligned} ds^2 \;&=\; -F(r)\,(dt-2\,N\,\cos\theta_1\,d\phi_1-2\,N\,\cos\theta_2\,d\phi_2)^2+\frac{dr^2}{F(r)}\\ &\quad +(r^2+N^2)\,(d\theta_1^2+\sin^2\theta_1\,d\phi_1^2+d\theta_2^2+\sin^2\theta_2\,d\phi_2^2)\,, \end{aligned} \tag{68}$$

where now

$$\begin{aligned} F(r)\;&=\;\frac{q^2\,(3r^2+N^2)}{(r^2+N^2)^4}\\ &\quad+\frac{1}{3\ell^2(r^2+N^2)^2}\left[\ell^2(-3N^4-6mr+6N^2r^2+r^4)-15N^6+45N^4r^2+15N^2r^4+3r^6\right]. \end{aligned}$$

Once again the correct background is found by setting $m=q=0$ but $N\neq 0$, and the energy is

$$E=12\,\frac{m}{G_6}. \tag{69}$$

In both cases, the electric charge q does not appear in the total energy just like in the case of ordinary Reissner-Nordström solution.

5. Higher curvature Gravity models

In flat backgrounds, the ghost freedom of low energy string theory requires the quadratic corrections to Einstein's gravity to be of the Gauss-Bonnet (GB) form, an argument that should carry over to the AdS backgrounds. Below we construct and compute the energy of various asymptotically (A)dS spaces that solve the generic Einstein plus quadratic gravity theories, particularly the Einstein-GB model [2, 29].

At quadratic order, the generic action is

$$I=\int d^Dx\sqrt{-g}\{\frac{1}{\kappa}R+\alpha R^2+\beta R_{\mu\nu}^2+\gamma(R_{\mu\nu\rho\sigma}^2-4R_{\mu\nu}^2+R^2)\} \tag{70}$$

In $D=4$, the GB part (γ terms) is a surface integral and plays no role in the equations of motion. In $D>4$, on the contrary, GB is the only viable term, since non-zero α,β produce ghosts [30]. Here $\kappa=2\Omega_{D-2}G_D$, where G_D is the D-dimensional Newton's constant [2].

After somewhat lengthy calculations we find the equations of motion

$$\begin{aligned} &\frac{1}{\kappa}(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R)+2\alpha R(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)+(2\alpha+\beta)(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R\\ +\;\;&2\gamma[RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}-R_{\mu\sigma}R_\nu{}^\sigma-\frac{1}{4}g_{\mu\nu}(R_{\tau\lambda\sigma\rho}^2-4R_{\sigma\rho}^2+R^2)]\\ +\;\;&\beta\Box(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R)+2\beta(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho}=\tau_{\mu\nu}. \end{aligned} \tag{71}$$

In the absence of matter, flat space is a solution of these equations; but more important is that (A)dS is also a solution [2]. The cosmological constant can be found using (71):

$$\begin{aligned} & \frac{1}{\kappa}(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}) + 2\alpha\bar{R}(\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}) \\ + \quad & (2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu)\bar{R} \\ + \quad & 2\gamma[\bar{R}\bar{R}_{\mu\nu} - 2\bar{R}_{\mu\sigma\nu\rho}\bar{R}^{\sigma\rho} + \bar{R}_{\mu\sigma\rho\tau}\bar{R}_\nu{}^{\sigma\rho\tau} \\ - \quad & 2\bar{R}_{\mu\sigma}\bar{R}_\nu{}^\sigma - \frac{1}{4}\bar{g}_{\mu\nu}(\bar{R}^2_{\tau\lambda\sigma\rho} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2)] \\ + \quad & \beta\bar{\Box}(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}) + 2\beta(\bar{R}_{\mu\sigma\nu\rho} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}_{\sigma\rho})\bar{R}^{\sigma\rho} = 0\,. \end{aligned} \tag{72}$$

The terms that have covariant derivatives will be zero. The other terms can be calculated one by one and they give us

$$-\frac{1}{2\Lambda\kappa} = \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \gamma\frac{(D-4)(D-3)}{(D-2)(D-1)}, \tag{73}$$

where $\Lambda \neq 0$ [2, 31]. Several comments are in order here: In the string-inspired Einstein-GB model ($\alpha = \beta = 0$ and $\gamma > 0$), only AdS background ($\Lambda < 0$) is allowed (the Einstein constant κ is positive in our conventions). String theory is known to prefer AdS to dS [2] and we can see why this is so in the uncompactified theory. Another interesting limit is the "traceless" theory ($D\alpha = -\beta$), which, in the absence of a γ term, does not allow constant curvature spaces unless the Einstein term is also dropped. For $D = 4$, the γ term drops out, and the pure quadratic theory allows (A)dS solutions with arbitrary Λ. For $D > 4$, (73) leaves a two-parameter set (say α, β) of allowed solutions [2].

Now we will linearize the equations of motion to first order in $h_{\mu\nu}$ and define the total energy-momentum tensor $T_{\mu\nu}$ as we did before.

$$\begin{aligned} T_{\mu\nu}(h) \quad = \quad & T_{\mu\nu}(\bar{g}) + \mathcal{G}^L_{\mu\nu}\left\{\frac{1}{\kappa} + \frac{4\Lambda D\alpha}{(D-2)} + \frac{4\Lambda\beta}{(D-1)} + \frac{4\Lambda\gamma(D-3)(D-4)}{(D-1)(D-2)}\right\} \\ & +(2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\right)R^L \\ & +\beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L\right) \\ & -2\Lambda^2 h_{\mu\nu}\left\{\frac{1}{2\Lambda\kappa} + \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \frac{\gamma(D-4)(D-3)}{(D-2)(D-1)}\right\}. \end{aligned} \tag{74}$$

One has $T_{\mu\nu}(\bar{g}) = 0$ and the last term also vanishes, yielding

$$\begin{aligned} T_{\mu\nu} \quad = \quad & \mathcal{G}^L_{\mu\nu}\left\{-\frac{1}{\kappa} + \frac{4\Lambda D}{(D-2)^2}(2\alpha + \frac{\beta}{(D-1)})\right\} \\ & +(2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\right)R^L \\ & +\beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L\right). \end{aligned} \tag{75}$$

This is a background conserved tensor ($\bar{\nabla}^\mu T_{\mu\nu} = 0$) as can be checked explicitly. An important aspect of (75) is the sign change of the $\frac{1}{\kappa}$ term relative to Einstein theory, due to the GB contributions. Hence in the Einstein-GB limit, we have $T_{\mu\nu} = -\mathcal{G}_{\mu\nu}{}^L/\kappa$, with the overall sign exactly opposite to that of the cosmological Einstein theory. However, this does not mean that E is negative there [2, 32].

There remains now to obtain a Killing energy expression from (75), namely, to write $\bar{\xi}_\nu T^{\mu\nu}$ as a surface integral. The first term is the usual AD piece (23). The term in the middle (which has the coefficient $2\alpha+\beta$), is easy to handle. First we take the indices up and than operate on this equation with a Killing vector, say $\bar{\xi}_\nu$,

$$\bar{\xi}^\mu \bar{\Box} R^L - \bar{\xi}^\nu \bar{\nabla}^\mu \bar{\nabla}_\nu R^L + \frac{2\Lambda}{(D-2)} \bar{\xi}^\mu R_L. \tag{76}$$

In the first term the covariant derivative must be taken outside to get surface terms:

$$\begin{aligned}
\bar{\xi}^\mu \bar{\nabla}_\alpha \bar{\nabla}^\alpha R_L &= \bar{\nabla}_\alpha (\bar{\xi}^\mu \bar{\nabla}^\alpha R_L) - (\bar{\nabla}_\alpha \bar{\xi}^\mu)(\bar{\nabla}^\alpha R_L) \\
&= \bar{\nabla}_\alpha (\bar{\xi}^\mu \bar{\nabla}^\alpha R_L) - \bar{\nabla}_\alpha (R_L \bar{\nabla}^\alpha \bar{\xi}^\mu) + R_L (\bar{\Box} \bar{\xi}^\mu) \\
&= \bar{\nabla}_\alpha \{\bar{\xi}^\mu \bar{\nabla}^\alpha R_L - R_L \bar{\nabla}^\alpha \bar{\xi}^\mu\} - \frac{2\Lambda}{(D-2)} R_L \bar{\xi}^\mu .
\end{aligned}$$

In the second term of (76), we can easily change the places of covariant and contravariant derivatives because of the Ricci scalar R_L, that is

$$\bar{\xi}^\nu \bar{\nabla}^\mu \bar{\nabla}_\nu R_L = \bar{\xi}^\nu \bar{\nabla}_\nu \bar{\nabla}^\mu R_L ,$$

and making the $\nu \longrightarrow \alpha$ substitution, we have

$$\begin{aligned}
\bar{\xi}^\alpha \bar{\nabla}_\alpha \bar{\nabla}^\mu R_L &= \bar{\nabla}_\alpha (\bar{\xi}^\alpha \bar{\nabla}^\mu R_L) - (\bar{\nabla}_\alpha \bar{\xi}^\alpha)(\bar{\nabla}^\mu R_L) \\
&= \bar{\nabla}_\alpha (\bar{\xi}^\alpha \bar{\nabla}^\mu R_L) ,
\end{aligned}$$

where $\bar{\nabla}_\alpha \bar{\xi}^\alpha$ is zero because of the Killing equation. Inserting these results into (76), the surface terms can be taken out

$$\begin{aligned}
&\bar{\xi}^\mu \bar{\Box} R^L - \bar{\xi}^\nu \bar{\nabla}^\mu \bar{\nabla}_\nu R^L + \frac{2\Lambda}{(D-2)} \bar{\xi}^\mu R_L \\
= \;& \bar{\nabla}_\alpha \{\bar{\xi}^\mu \bar{\nabla}^\alpha R_L - R_L \bar{\nabla}^\alpha \bar{\xi}^\mu\} - \frac{2\Lambda}{(D-2)} R_L \bar{\xi}^\mu + \frac{2\Lambda}{(D-2)} R_L \bar{\xi}^\mu - \bar{\nabla}_\alpha (\bar{\xi}^\alpha \bar{\nabla}^\mu R_L) \\
= \;& \bar{\nabla}_\alpha \{\bar{\xi}^\mu \bar{\nabla}^\alpha R_L - \bar{\xi}^\alpha \bar{\nabla}^\mu R_L + R_L \bar{\nabla}^\mu \bar{\xi}^\alpha\} .
\end{aligned}$$

The last term in (75) can be written as a surface term plus extra terms:

$$\begin{aligned}
\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} &= \bar{\xi}_\nu \bar{\nabla}_\alpha \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} \\
&= \bar{\nabla}_\alpha \{\bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu}\} - (\bar{\nabla}_\alpha \bar{\xi}_\nu)(\bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu}) ,
\end{aligned}$$

where we have put the Killing vector inside the covariant derivative. In the second term we can freely move the α indices and afterwards, we can also take terms inside the covariant derivative with an extra term. Hence, we get

$$\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} = \bar{\nabla}_\alpha \{\bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu\} + \mathcal{G}_L^{\mu\nu} \bar{\Box} \bar{\xi}_\nu .$$

Now we can add and subtract the terms $\bar{\nabla}_\alpha \{\bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu}\}$ and $\bar{\nabla}_\alpha \{\mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu\}$

$$\begin{aligned}
\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} &= \bar{\nabla}_\alpha \{\bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu + \mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu\} \\
&\quad + \mathcal{G}_L^{\mu\nu} \bar{\Box} \bar{\xi}_\nu + \bar{\nabla}_\alpha \{\bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu}\} - \bar{\nabla}_\alpha \{\mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu\}.
\end{aligned}$$

If we expand the last two terms we can see that: (i) When the covariant derivative hits on the Killing vector $\bar{\xi}_\nu$, it will be zero in the first one with the use of Killing vector equation, because α and ν are symmetric in $\mathcal{G}_L^{\alpha\nu}$. (ii) With the help of Bianchi identity, the term $(\bar{\nabla}_\alpha \mathcal{G}_L^{\alpha\nu})(\bar{\nabla}^\mu \bar{\xi}_\nu)$ is zero. Hence we are left with

$$\begin{aligned}
\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} &= \bar{\nabla}_\alpha \{\bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu + \mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu\} \\
&\quad + \mathcal{G}_L^{\mu\nu} \bar{\Box} \bar{\xi}_\nu + \bar{\xi}_\nu \bar{\nabla}_\alpha \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\alpha\nu} \bar{\nabla}_\alpha \bar{\nabla}^\mu \bar{\xi}_\nu .
\end{aligned} \tag{77}$$

We can write $\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu}$ as a surface term. Collecting everything, the final form of the conserved charges for the generic quadratic theory reads

$$\begin{aligned} Q^\mu(\bar{\xi}) &= \{-\frac{1}{\kappa} + \frac{8\Lambda}{(D-2)^2}(\alpha D + \beta)\} \int d^{D-1}x \sqrt{-\bar{g}}\, \bar{\xi}_\nu \, \mathcal{G}_L^{\mu\nu} \\ &+ (2\alpha + \beta) \int dS_i \sqrt{-\bar{g}} \{\bar{\xi}^\mu \bar{\nabla}^i R_L - \bar{\xi}^i \bar{\nabla}^\mu R_L + R_L \bar{\nabla}^\mu \bar{\xi}^i\} \\ &+ \beta \int dS_i \sqrt{-\bar{g}} \{\bar{\xi}_\nu \bar{\nabla}^i \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{i\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^i \bar{\xi}_\nu + \mathcal{G}_L^{i\nu} \bar{\nabla}^\mu \bar{\xi}_\nu\}. \end{aligned} \tag{78}$$

Now let us compute the energy of an asymptotically SdS geometry that might be a solution to our generic model. Should such a solution exist, we only require its asymptotic behavior to be

$$h_{00} \approx + \left(\frac{r_0}{r}\right)^{D-3}, \quad h^{rr} \approx + \left(\frac{r_0}{r}\right)^{D-3} + O(r_0^{2(D-2)}). \tag{79}$$

It is easy to see that for asymptotically SdS spaces the second and the third lines of (78) do not contribute, since for any Einstein space, to linear order

$$R^L_{\mu\nu} = \frac{2\Lambda}{D-2} h_{\mu\nu}, \tag{80}$$

which in turn yields $R_L = \bar{g}^{\mu\nu} R^L_{\mu\nu} - [2\Lambda/(D-2)]h = 0$ and thus $\mathcal{G}^L_{\mu\nu} = 0$ in the asymptotic region. Therefore the total energy of the full (α, β, γ) system, for geometries that are asymptotically SdS, is given only by the first term in (78),

$$E_D = \left\{-1 + \frac{8\Lambda\kappa}{(D-2)^2}(\alpha D + \beta)\right\} \frac{(D-2)}{4G_D} r_0^{D-3}, \quad D > 4, \tag{81}$$

where γ is implicitly assumed not to vanish. For $D = 4$, equivalently from (74), it reads (for models with an explicit Λ)

$$E_4 = \{1 + 2\Lambda\kappa(4\alpha + \beta)\} \frac{r_0}{2G_4}. \tag{82}$$

In $D = 3$, the GB density vanishes identically and the energy expression has the same form of the $D = 4$ model, with the difference that r_0 comes from the metric (32) [2].

From (81), the asymptotically SdS solution seemingly has negative energy, in the Einstein-GB model:

$$E = -\frac{(D-2)}{4G_D} r_0^{D-3}. \tag{83}$$

While this is, of course, correct in terms of the usual SdS signs, their exact form is [32]

$$ds^2 = g_{00} dt^2 + g_{rr} dr^2 + r^2 d\Omega_{D-2}, \tag{84}$$

$$\begin{aligned} -g_{00} = g_{rr}^{-1} &= 1 + \frac{r^2}{4\kappa\gamma(D-3)(D-4)} \\ &\times \left(1 \pm \left[1 + 8\gamma(D-3)(D-4)\frac{r_0^{D-3}}{r^{D-1}}\right]^{\frac{1}{2}}\right). \end{aligned} \tag{85}$$

Note that there is a branching here, with qualitatively different asymptotics: Schwarzschild and Schwarzschild-AdS,

$$\begin{aligned} -g_{00} &= 1 - \left(\frac{r_0}{r}\right)^{D-3}, \\ &= 1 + \left(\frac{r_0}{r}\right)^{D-3} + \frac{r^2}{\kappa\gamma(D-3)(D-4)}. \end{aligned} \tag{86}$$

[Here we restored γ, using $\kappa\gamma(D-3)(D-4) = -l^2$.] The first solution has the usual positive (for positive r_0 of course) ADM energy $E = +(D-2)r_0^{D-3}/4G_D$, since the GB term does not contribute when expanded around flat space. On the other hand, the second solution which is asymptotically SdS, has the wrong sign for the "mass term". However, to actually compute the energy here, one needs our energy expression (78), and not simply the AD formula which is valid only for cosmological Einstein theory. Now from (86), we have

$$h_{00} \approx -\left(\frac{r_0}{r}\right)^{D-3}, \quad h^{rr} \approx -\left(\frac{r_0}{r}\right)^{D-3} + O(r_0^{2(D-2)}), \tag{87}$$

whose sign is opposite to that of the usual SdS. This sign just compensates the flipped sign in the energy definition, so the energy (81) reads $E = (D-2)r_0^{D-3}/4G_D$ and the AdS branch, just like the flat branch, has positive energy, after the GB effects are taken into account also in the energy definition. Thus, for every Einstein-GB external solution, energy is positive and AdS vacuum is stable [2, 32].

6. Conclusion

In this review, we have defined the energy of generic Einstein plus cosmological term plus quadratic gravity theories in generic D dimensions, for both asymptotically flat and (A)dS spaces. We have computed the masses and angular momenta of various solutions, including D dimensional Kerr-AdS solution.

Our construction is based on the existence of background Killing vectors. For the background, we have chosen either constant curvature (AdS) or flat spacetimes. In our formalism, the background always has zero charge. We would like to stress that this review does not present new material but simply extend some of the work we have done recently. The interested reader is referred to the original material cited in our references.

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