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Negative mass solitons in gravity

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We first reconstruct the conserved (Abbott-Deser) charges in the spin-connection formalism of gravity for asymptotically (Anti)-de Sitter spaces, and then compute the masses of the AdS soliton and the recently found Eguchi-Hanson solitons in generic odd dimensions, unlike the previous result obtained for only five dimensions. These solutions have negative masses compared to the global AdS or AdS/Z_p spacetimes. As a separate note, we also compute the masses of the recent even dimensional Taub-NUT-Reissner-Nordström metrics.

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I. INTRODUCTION

Energy definition in theories with gravity has been a thorny issue since the inception of General Relativity. Even though Einstein's equation relates local properties of geometry to the local properties of matter, when integrated it, nevertheless, requires one to express the properties (such as mass) of gravitating matter in terms of diffeomorphism invariant geometric quantities. In a theory without gravity, such as quantum field theory in flat spacetime, obtaining the conserved charges, a la Noether, would be a straightforward task. However, Noether's method, as employed by Komar [1], leads to certain ambiguities with gravity; such as assigning different normalization factors—to match the weak field Newtonian limits—for the mass and angular momenta of asymptotically flat black hole spacetimes. [Unlike the conserved charges of isolated local objects in gravity-free theories, one can only talk about the *total* energy of a spacetime since gravity cannot be confined to a region and, as long as diffeomorphism invariance is required, one has to talk about the energy of a whole spacetime as opposed to a finite domain.]

There are remedies for Komar's method but the problem is that there are simply ''too many'' different ones for various spacetimes. There is the classic work of Arnowitt, Deser and Misner (ADM) [2] which defines a Hamiltonian for asymptotically flat spacetimes. The later work of Regge and Teitelboim [3] introduces the conserved charges as boundary terms that are required for a proper variational formulation of the problem. For spacetimes which are asymptotically Anti-de-Sitter (AdS) (or asymptotically locally AdS), one can find plenty of energy definitions given e.g. by Abbott-Deser [4,5], Ashtekar-Magnon [6], Hawking-Horowitz [7], Aros *et al.* [8], Cai-Cao [9], Henneaux-Teitelboim [10], Henningson-Skenderis [11] and Balasubramanian-Kraus [12]. [See also Barnich *et al.* [13,14] for conserved charges in generic gauge theories, including gravity.] Unfortunately, each one of these methods have strong and weak points. A recent detailed comparison of some of these definitions was nicely carried out by Hollands *et al.* [15]. Therefore, when an ''interesting'' solution for a gravity theory is found, one wonders about the conserved charges, and more specifically, the mass of that solution computed by the above methods. Although these methods frequently agree, one can easily find examples where they do not [See e.g. [16] in which it was shown that the existence of long-range scalar fields leads to discrepancies between the Abbott-Deser and Ashtekar-Magnon definitions for certain spacetimes.]

Recently Clarkson and Mann [17] found new solitons in cosmological spacetimes that have quite interesting properties: They resemble the Eguchi-Hanson [18] metrics with AdS/Z_p asymptotics. For the case of a negative cosmological constant, these solutions have lower energy than the global AdS/Z_p spacetime. Clarkson and Mann computed the energy of the 5-dimensional solution using the boundary counterterm method of Henningson and Skenderis [11]. The authors of [17] also claimed that these solutions have the lowest energy in their asymptotic class. In fact, by now, we are used to such novel properties of AdS spacetimes: Horowitz and Myers [19] provided us with the first example of a negative mass soliton, called the ''AdS soliton''. These negative energy solutions do not cause any instabilities, as in the case of a scalar field of negative mass-squared satisfying the Breitenlohner-Freedman [20] bound. The stability of negative mass solitons in the context of the AdS/CFT correspondence is expected since the field theory vacuum is stable.

In this paper, we shall compute the masses of both the AdS soliton and the recently found Eguchi-Hanson (EH) solitons using the Abbott-Deser [4] procedure which can be quite easily generalized to higher curvature models of gravity [5]. We would like to stress that, unlike the boundary counterterm approach which works for a given fixed

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dimension, our method applies to generic dimensions and here we will compute the masses for arbitrary (odd) dimensions. However, before computing the charges, we will first reconstruct the conserved charges for cosmological Einstein's theory formulated with the spin connection and the vielbein instead of the metric. This is a straightforward, yet a tedious task. Whenever fermionic fields are to be taken into account, such as in supergravity theories, one has to use the "first order" spin-connection formulation. We believe that this provides an important motivation as to why conserved charges in the latter formalism needs to be worked out.

Before we move on to the bulk of the paper, we would like to mention that observations point out that the Universe might have a small positive cosmological constant. For this very exciting possibility, in principle, one would like to study various properties, such as conserved charges, stability, *etc.* of de Sitter (dS) spacetimes as opposed to the AdS spacetimes. But ironically, most of the recent theoretical progress (such as the remarkable AdS/CFT dictionary) has been on spacetimes with a negative cosmological constant. Global properties of the latter has little, if any, resemblance to the former: Therefore, it is not exactly clear how one would make use of the enormous amount of information gained in negatively curved spacetimes. This is a serious challenge but it does not deter us from studying the AdS spacetimes. In fact, for "small" objects (black holes and so on) that do not change the location of the cosmological horizon, we want to emphasize that, our formulas will define mass within the cosmological horizon in de Sitter spacetimes. Moreover they are also easily modifiable to apply to the higher curvature models, such as the Gauss-Bonnet theory.

II. CONSERVED CHARGES IN ASYMPTOTICALLY AdS SPACETIMES

In this section, we shall first briefly recapitulate the construction carried out in Abbott-Deser [4] and Deser-Tekin [5] papers and then redrive the surface integrals for conserved charges in cosmological Einstein theory using the modern language of differential forms. As mentioned above, the charge definition that we are about to present is neither unique nor, in general, in agreement with some other definitions for all spacetimes. However, we would like to point out that our definition is quite intuitive and physical: the background spacetime (the global AdS) has zero energy and the asymptotically AdS spacetimes have energy measured with respect to the background. In some sense, an observer sitting at the boundary of the spacetime (that is, at the spatial infinity), sees a black hole as a perturbation to the background spacetime. Let us formulate this idea by splitting the metric into a background plus a perturbation:

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $g_{\mu\nu}$ is a solution to a certain gravity theory coupled to matter sources. Note that this theory need not be Einstein's theory: It could be a complicated higher curvature gravity model. What we require from this model is that, it either come from a proper local Lagrangian or it be endowed with the Bianchi identities and covariant conservation of the matter tensor (or identically, the left hand side—*i.e.* the geometry part—of the equations of motions). In what follows "barred" quantities refer to the background spacetime that is a solution to the equations of motion without a source term. We assume that there are background Killing vectors (to be able to define energy, one of these vectors has to be a timelike vector everywhere)

$$\bar{\nabla}_\mu \bar{\xi}^{(a)}_\nu + \bar{\nabla}_\nu \bar{\xi}^{(a)}_\mu = 0. \tag{2}$$

Having the Killing equation at our disposal, we can construct partially conserved vector currents out of the covariantly conserved tensor currents of the linearized equations. For example, this procedure (worked out in detail in [5]) leads to the following conserved charges in cosmological Einstein theory

$$\begin{aligned} Q^\mu(\bar{\xi}) = \frac{1}{4\Omega_{D-2}G_D} \int_{\partial M} dS_i \{ & \bar{\xi}_\nu \bar{\nabla}^\mu h^{i\nu} - \bar{\xi}_\nu \bar{\nabla}^i h^{\mu\nu} \\ & + \bar{\xi}^\mu \bar{\nabla}^i h - \bar{\xi}^i \bar{\nabla}^\mu h + h^{\mu\nu} \bar{\nabla}^i \bar{\xi}_\nu - h^{i\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \\ & + \bar{\xi}^i \bar{\nabla}_\nu h^{\mu\nu} - \bar{\xi}^\mu \bar{\nabla}_\nu h^{i\nu} + h \bar{\nabla}^\mu \bar{\xi}^i \}, \end{aligned} \tag{3}$$

where $h = h_{\mu\nu}\bar{g}^{\mu\nu}$ and the gravitational charge has been normalized by the D-dimensional Newton's constant G_D and the solid angle of a $(D-2)$-sphere S^{D-2}. Recently, this formula was successfully applied [21] to the D-dimensional Kerr-AdS black holes [22] and a modified version of it [23] was used to calculate the charges of the BTZ black hole [24] and the charges of the only known supersymmetric solution to the topologically massive gravity [25] in $D = 3$ [26]. If there are higher curvature terms present, the construction gets modified as worked out in detail in [5] and outlined below.

We now turn on to a detailed computation of the conserved gravitational charges formulated with the spin connection and the vielbein. As already mentioned in the introduction, such a formulation is forced on us in the presence of fermions, for example, in any supergravity theory. [As a side note, recall that if the vielbein is assumed to be invertible (nondegenerate), then the spin-connection formulation is equivalent to the metric formulation.]

Consider a generic gravity theory coupled to a covariantly conserved bounded matter source which is described by the following "Einstein" equations:

$$G_a + \Lambda \star e_a = \kappa T_a. \tag{4}$$

Here G_a is the 'Einstein $(D-1)$-form' of a local *generic* gravity action, $\star$ is the Hodge star operator and κ is the relevant "coupling constant" of the model under investigation. Suppose now that the metric tensor g

$$g = \eta_{ab} e^a \otimes e^b,$$

is decomposed such that the "full" orthonormal coframe 1-forms e^a can be written as the sum of a "background" orthonormal coframe $\bar{e}^a$ [which satisfies (4) for $T_a = 0$] plus a "deviation" piece as

$$e^a \equiv \bar{e}^a + \varphi^a{}_b \bar{e}^b, \tag{5}$$

where the 0-forms $\varphi^a{}_b$ are assumed to vanish sufficiently rapidly at "infinity". [Note that the decomposition described by (5) is always possible given a metric tensor g and a choice for the "background" coframes $\bar{e}^a$, since one can always write $e^a = \bar{e}^a + \psi^a{}_\mu dx^\mu$ and $dx^\mu = \bar{E}^\mu{}_b \bar{e}^b$, for some 0-forms $\psi^a{}_\mu$ and $\bar{E}^\mu{}_b$, which means $\varphi^a{}_b = \psi^a{}_\mu \bar{E}^\mu{}_b$ in (5).] One can now separate the field Eqs. (4) into a part linear in $\varphi^a{}_b$ plus all the remaining nonlinear parts so that, one obtains

$$\bar{G}_a(\varphi^b{}_c) = \kappa \tau_a,$$

the "linearized" version of the field Eqs. (4). Here $\bar{G}_a(\varphi^b{}_c)$ is a $(D-1)$-form that involves only terms linear in the deviation parts $\varphi^b{}_c$ and depends only on the background coframes $\bar{e}^a$ (and, of course, the differential geometric structures that they define); the $(D-1)$-form τ_a naturally contains all the nonlinear terms in $\varphi^b{}_c$ plus the contributions from the original matter source T_a.

It can be shown that due to the background Bianchi identity and the background gauge invariance, there exists a set, denoted by the index I, of Killing vectors $\bar{\xi}_a{}^{(I)}$

$$\bar{D}_a \bar{\xi}_b{}^{(I)} + \bar{D}_b \bar{\xi}_a{}^{(I)} = 0, \tag{6}$$

for the background geometry described by $\bar{e}^a$. Here $\bar{D}_a \equiv \bar{\iota}_a \bar{D}$; $\bar{\iota}_a$ denotes the interior product operator with respect to a "background" frame vector that acts on the space of forms and creates a $(p-1)$-form out of a p-form so that, *e.g.* $\bar{\iota}_b \bar{e}^a = \delta_b{}^a$; $\bar{D}$ denotes the covariant derivative operator with respect to the Levi-Civita connection 1-forms $\bar{\omega}^a{}_b$ of the background coframes that satisfy the Cartan structure equations $d\bar{e}^a + \bar{\omega}^a{}_b \wedge \bar{e}^b = 0$. Since $\bar{D}\tau^a = 0$ by the background Bianchi identity, it readily follows that one also has

$$\bar{D}(\tau^c \bar{\xi}_c{}^{(I)}) = d(\tau^c \bar{\xi}_c{}^{(I)}) = 0.$$

However, using the fact that the torsion 2-form vanishes, *i.e.* $\bar{D}\bar{e}_a = 0$, and defining $\tau^c = \tau^{ca} \bar{\star} \bar{e}_a$ for some 0-forms τ^{ca}, one can come up with a conserved density current that leads to the following conserved Killing charges

$$Q^a(\bar{\xi}^{(I)}) = \int_M \tilde{\star} 1 \tau^{ca} \bar{\xi}_c{}^{(I)} = \int_{\partial M} dS_i q^{ai(I)}. \tag{7}$$

Here M is a spatial $(D-1)$-dimensional hypersurface, $\tilde{\star}1$ is the oriented "volume" element of M, ∂M denotes its $(D-2)$-dimensional boundary, we use dS_i to denote the corresponding "area" element of ∂M, and $q^{ai(I)}$ is obtained from $\bar{G}_a(\varphi^b{}_c)$ whose explicit form depends on the theory being studied. Here the index i ranges over $1, 2, \ldots, D-2$ and we have used Stokes' theorem (and the usual accompanying assumptions of it) to obtain this final form for the Killing charge. [Note that to apply the Stokes' theorem, it is of course necessary to write $\tau^{ca}\bar{\xi}_c{}^{(I)} = \bar{D}_c q^{ac(I)}$, which is the tricky part but holds for all "physically reasonable" theories that we know.]

Let us be more explicit now and consider the most "relevant" example of the D-dimensional cosmological Einstein theory for which the vacuum equations read

$$-\frac{1}{2} R^{ab} \wedge \star e_{abc} + \Lambda \star e_c = 0. \tag{8}$$

Here e_{abc} is a shorthand notation for $e_a \wedge e_b \wedge e_c$ and we use analogous expressions for e_{ab}, etc. Vacuum equations are solved by a space of constant curvature which satisfies

$$\begin{aligned} \bar{R}_{abcd} &= \frac{2\Lambda}{(D-1)(D-2)}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}), \\ \bar{R}_{ab} &= \frac{1}{2}\bar{R}_{abcd}\bar{e}^{cd} = \frac{2\Lambda}{(D-1)(D-2)}\bar{e}_{ab}, \\ \bar{R} &= \bar{\iota}_b \bar{\iota}_a \bar{R}^{ab} = \frac{2\Lambda D}{D-2}. \end{aligned} \tag{9}$$

The "linearization" process of (8) coupled to a matter source in the sense described above involves the use of many nontrivial identities and somewhat complicated calculations. We present these technical derivations in Appendix A and proceed with the explicit form of the first integrand in (7) which reads

$$\begin{aligned} \bar{\xi}_c \tau^{ca} = {} & \left(-\bar{D}_c \bar{D}^b \varphi^c{}_b + \bar{D}_c \bar{D}^c \varphi^b{}_b + \frac{2\Lambda}{D-1}\varphi^c{}_c\right)\bar{\xi}^a \\ & - \frac{2\Lambda}{D-1}\bar{\xi}_c \varphi^{ac} - \bar{\xi}_c \bar{D}^c \bar{D}^a \varphi^b{}_b + \bar{\xi}_c \bar{D}^c \bar{D}^b \varphi^a{}_b \\ & - \bar{\xi}_c \bar{D}_b \bar{D}^b \varphi^{ac} + \bar{\xi}_c \bar{D}_b \bar{D}^a \varphi^{bc} \end{aligned} \tag{10}$$

for a given Killing vector $\bar{\xi}_c$ of the "background". Here, and in what follows, we suppress the further use of the index I which labels the Killing vectors of the background geometry.

The nontrivial task to fulfill now is to put everything on the right hand side of (10) in the form $\bar{D}_c$(something) and the details of this are given in appendix B. The outcome of this procedure is

$$\begin{aligned} \bar{\xi}_c \tau^{ca} = \bar{D}_c(& -\bar{\xi}^a \bar{D}^b \varphi^c{}_b + \varphi^{bc}\bar{D}_b \bar{\xi}^a - \varphi^b{}_b \bar{D}^c \bar{\xi}^a \\ & + \bar{\xi}^a \bar{D}^c \varphi^b{}_b - \bar{\xi}^c \bar{D}^a \varphi^b{}_b + \bar{\xi}^c \bar{D}^b \varphi^a{}_b \\ & - \bar{\xi}_b \bar{D}^c \varphi^{ab} + \varphi^{ab}\bar{D}^c \bar{\xi}_b + \bar{\xi}_b \bar{D}^a \varphi^{cb}), \end{aligned} \tag{11}$$

which explicitly yields the following conserved Killing charge corresponding to (7)

$$Q^a(\bar{\xi}) = \frac{1}{4\Omega_{D-2}G_D}\int_{\partial M} dS_i(-\bar{\xi}^a\bar{D}^b\varphi^i{}_b + \varphi^{bi}\bar{D}_b\bar{\xi}^a - \varphi^b{}_b\bar{D}^i\bar{\xi}^a + \bar{\xi}^a\bar{D}^i\varphi^b{}_b - \bar{\xi}^i\bar{D}^a\varphi^b{}_b + \bar{\xi}^i\bar{D}^b\varphi^a{}_b - \bar{\xi}_b\bar{D}^i\varphi^{ab} + \varphi^{ab}\bar{D}^i\bar{\xi}_b + \bar{\xi}_b\bar{D}^a\varphi^{ib}). \qquad (12)$$

As expected, this is similar in form to the metric formulation (3), but the details were needed to be worked out carefully since the spin-connection and the metric formulation are quite distinct in spirit. Had we considered a generic higher curvature model in the spin-connection formulation, the charges would have been modified along the lines of [5]. Here, we will not do that computation, but simply say that, for quadratic gravity models, such as the Gauss-Bonnet or any R^2 theory, a nontrivial factor (depending on the coefficients of the higher curvature terms and the cosmological constant) will multiply the charge in (12).

III. COMPUTATION OF THE CHARGES FOR THE SOLITONS

A. AdS soliton

Our first example is the "AdS Soliton" of Horowitz-Myers [19]

$$ds^2 = \frac{r^2}{\ell^2}\left[\left(1 - \frac{r_0^{p+1}}{r^{p+1}}\right)d\tau^2 + \sum_{i=1}^{p-1}(dx^i)^2 - dt^2\right] + \left(1 - \frac{r_0^{p+1}}{r^{p+1}}\right)^{-1}\frac{\ell^2}{r^2}dr^2, \qquad (13)$$

which was obtained by the double analytic continuation of a near extremal p-brane solution. Here x^i $(i = 1, \ldots, p-1)$ and the t variables denote the coordinates on the "brane" and $r \geq r_0$. To avoid a conical singularity at $r = r_0$, τ necessarily has a period $\beta = 4\pi\ell^2/(r_0(p+1))$. Its energy was computed in [19] using the method of [7]. Here we compute the energy using the method described so far. The background ($r_0 = 0$) is the usual globally AdS spacetime in the horospherical coordinates, with the timelike Killing vector

$$\bar{\xi}^\mu = (-1, 0, \ldots, 0). \qquad (14)$$

Defining the metric perturbation as outlined above and carrying out the integrations, we have

$$E = -\frac{V_{D-3}\pi}{(D-1)\Omega_{D-2}G_D}\frac{r_0^{D-2}}{\ell^{D-2}}, \qquad (15)$$

where V_{D-3} is the volume of the compact dimensions. Upto trivial charge normalizations, our result matches that of [19], which uses the energy definition of Hawking-Horowitz [7].

B. Eguchi-Hanson solitons

Recently, Clarkson and Mann [17] found very interesting solutions to the *odd* dimensional cosmological (for both signs) Einstein equations. These solutions resemble the even dimensional Eguchi-Hanson metrics [18]—thus the name Eguchi-Hanson solitons—and asymptotically approach AdS/Z_p, where $p \geq 3$. As shown in [17], these solutions have lower energy compared to the global AdS spacetimes (or the global AdS/Z_p spacetimes). The energies of these solutions (for the case of 5 dimensions) were computed in [17] with the help of the boundary counterterm method [11,12]. It is important to note that boundary counterterm method needs to be worked out for a given fixed dimension. Here, we use the prescription outlined in the previous section and find the energy of the EH solitons for generic odd dimensions. For a detailed description of the metrics, we refer the reader to [17]. We simply quote their result: the EH soliton reads

$$ds^2 = -g(r)dt^2 + \left(\frac{2r}{D-1}\right)^2 f(r)\left[d\psi + \sum_{i=1}^{(D-3)/2}\cos\theta_i d\phi_i\right]^2 + \frac{dr^2}{g(r)f(r)} + \frac{r^2}{D-1}\sum_{i=1}^{(D-3)/2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \qquad (16)$$

and the metric functions are given by

$$g(r) = 1 \mp \frac{r^2}{\ell^2}, \qquad f(r) = 1 - \left(\frac{a}{r}\right)^{D-1}. \qquad (17)$$

In the AdS case, to remove the stringlike singularity at $r = a$, one finds that ψ has a period $4\pi/p$ and there is a constraint on the parameter a:

$$a^2 = \ell^2\left(\frac{p^2}{4} - 1\right). \qquad (18)$$

The background is obtained simply by setting $a = 0$ in (17). The details of the energy (the charge for $\bar{\xi}^\mu = (-1, 0, \ldots, 0)$) computation is rather lengthy and not particularly illuminating to present here. Instead, we will only write down our result. For convenience, we define

$$E(\bar{\xi}) \equiv \frac{1}{4\Omega_{D-2}G_D}\int_{\partial M} dS_r \mathcal{E}(\bar{\xi}),$$

and only present $\mathcal{E}(\bar{\xi})$, in the $r \to \infty$ limit:

$$\lim_{r\to\infty}\mathcal{E}(\bar{\xi}) = -\frac{2a^{D-1}}{\ell^2(D-1)^{(D-1)/2}}\prod_{i=1}^{(D-3)/2}\sin\theta_i. \qquad (19)$$

After the angular integrations are carried out, one obtains the energy of the EH soliton in generic odd dimensions

$$E = -\frac{(4\pi)^{(D-1)/2}a^{D-1}}{p\ell^2(D-1)^{(D-1)/2}\Omega_{D-2}G_D}. \qquad (20)$$

Specifically, when $D = 5$, one finds

$$E = -\frac{a^4}{4p\ell^2 G_5}. \tag{21}$$

We note that this result differs from that of [17] in two respects: one of which is a trivial numerical factor that can be attributed to normalization of the conserved charges (12); the second, and the more important one, is the presence of an additive constant which is exactly equal to the energy of the AdS/Z_p spacetime. Recall that in the formalism we use, the background always has zero energy, unlike the boundary counterterm method for which it has a finite energy.

IV. CONCLUSIONS

In this paper, we have constructed the conserved charges for asymptotically AdS spacetimes using the spin connection and the vielbein formalism of gravity and then computed the gravitational energies of the AdS soliton and the recently found EH solitons. For the latter, our method provided us with a computation of the masses for generic odd dimensions, unlike the boundary counterterm method which was employed only for $D = 5$. These solitons all have lower energies than the global AdS or the global AdS/Z_p spacetimes.

We would like to stress that the Abbott-Deser [4,5] method, which we used here, is a powerful tool that can have a wide range of applications. For example, one can use it to compute the masses of the new charged solutions [27–29] in AdS spacetimes that have nontrivial topology. Here, we only consider two examples that were presented in [27]. In $D = 4$, the "Taub-NUT-Reissner-Nordström" solution reads

$$\begin{aligned} ds^2 &= -F(r)(dt - 2N\cos\theta d\phi)^2 + \frac{dr^2}{F(r)} \\ &\quad + (r^2 + N^2)(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{22}$$

where N is the nut charge and

$$F(r) = \frac{r^4 + (\ell^2 + 6N^2)r^2 - 2m\ell^2 r - 3N^4 + \ell^2(q^2 - N^2)}{\ell^2(r^2 + N^2)}. \tag{23}$$

To find the energy of this solution, the correct background (that has zero energy) needs to be carefully chosen. If we naively set $m = q = 0$ and the nut charge $N = 0$, then the energy of the solution with nonzero m, q, N diverges. This is to be expected since $N = 0$ solution is *not* in the same topological class as that of the $N \neq 0$ solutions. The background has to be chosen as $m = q = 0$ but $N \neq 0$ as was shown by Deser-Soldate [30] in the case of the (asymptotically locally flat) Kaluza-Klein monopole. In the light of these arguments, using (12) one gets

$$E = \frac{m}{G_4}. \tag{24}$$

In $D = 6$, the metric, for the details of which we refer to [27], reads

$$\begin{aligned} ds^2 &= -F(r)(dt - 2N\cos\theta_1 d\phi_1 - 2N\cos\theta_2 d\phi_2)^2 \\ &\quad + \frac{dr^2}{F(r)} + (r^2 + N^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 \\ &\quad + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \end{aligned} \tag{25}$$

where now

$$\begin{aligned} F(r) = \frac{q^2(3r^2+N^2)}{(r^2+N^2)^4} + \frac{1}{3\ell^2(r^2+N^2)^2}[\ell^2(-3N^4 - 6mr \\ + 6N^2r^2 + r^4) - 15N^6 + 45N^4r^2 + 15N^2r^4 + 3r^6]. \end{aligned}$$

Once again the correct background is found by setting $m = q = 0$ but $N \neq 0$, and the energy is

$$E = 12\frac{m}{G_6}. \tag{26}$$

In both cases, the electric charge q does not appear in the total energy just like in the case of ordinary Reissner-Nordström solution.

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APPENDIX A: THE DERIVATION OF (10)

In this Appendix, we present the technical calculations that lead to (10).

The interior product operator satisfies $\iota_b e^a = \delta_b{}^a$, which implies that $\bar{\iota}_b$ is related to ι_b by

$$\iota_b = \bar{\iota}_b - \varphi_b{}^c \bar{\iota}_c \tag{A1}$$

upto terms first order in $\varphi^a{}_b$. Moreover, substituting (5) in the defining relation of the Hodge operator

$$\begin{aligned} \star e^a &\equiv \frac{1}{(D-1)!}\epsilon^a{}_{bc\ldots d} e^b \wedge e^c \wedge \ldots \wedge e^d \\ &\equiv \frac{1}{(D-1)!}\epsilon^a{}_{bc\ldots d} e^{bc\ldots d}, \end{aligned}$$

yields

$$\begin{aligned} \star e^a &= \bar{\star}\bar{e}^a + \frac{1}{(D-2)!}\epsilon^a{}_{bc\ldots d}\varphi^b{}_p \bar{e}^{pc\ldots d} \\ &= \bar{\star}\bar{e}^a + \varphi^b{}_p \bar{e}^p \wedge \bar{\star}\bar{e}^a{}_b \end{aligned}$$

in terms of the Hodge operator $\bar{\star}$ of the "background". This identity can also be generalized to

$$\star e^{abc} = \bar{\star}\bar{e}^{abc} + \varphi^d{}_p \bar{e}^p \wedge \bar{\star}\bar{e}^{abc}{}_d$$

in a straightforward fashion. When the Cartan structure equations $De^a = de^a + \omega^a{}_b \wedge e^b = 0$ are solved for the Levi-Civita connection 1-forms, one finds

$$\omega^a{}_b = \frac{1}{2}(\iota_b de^a - \iota^a de_b + e^c(\iota^a \iota_b de_c)).$$

Since $de^a = d\bar{e}^a + d\varphi^a{}_b \wedge \bar{e}^b + \varphi^a{}_b d\bar{e}^b$, one obtains (using (A1) and (5)) that

$$\iota_b de_c = \bar{\iota}_b d\bar{e}_c + (\bar{\iota}_b d\varphi_{ck})\bar{e}^k - d\varphi_{cb} + \varphi_{ck}(\bar{\iota}_b d\bar{e}^k) - \varphi_b{}^k(\bar{\iota}_k d\bar{e}_c)$$

and

$$\iota^a \iota_b de_c = \bar{\iota}^a \bar{\iota}_b d\bar{e}_c + \bar{\iota}_b d\varphi_c{}^a - \bar{\iota}^a d\varphi_{cb} + \varphi_{ck}(\bar{\iota}^a \bar{\iota}_b d\bar{e}^k) - \varphi_b{}^k(\bar{\iota}^a \bar{\iota}_k d\bar{e}_c) - \varphi^{ak}(\bar{\iota}_k \bar{\iota}_b d\bar{e}_c)$$

up to first order ‘‘deviation‘‘ terms. Keeping in mind that the ‘‘background‘‘ Levi-Civita connection 1-forms satisfy $\bar{D}\bar{e}^a = d\bar{e}^a + \bar{\omega}^a{}_b \wedge \bar{e}^b = 0$, and using the fact that $\bar{D}\varphi^a{}_c = d\varphi^a{}_c + \bar{\omega}^a{}_k \varphi^k{}_c - \bar{\omega}^k{}_c \varphi^a{}_k$, one thus finds

$$\omega^a{}_b = \bar{\omega}^a{}_b + \bar{e}^c[\bar{D}_b \varphi^a{}_c - \bar{D}^a \varphi_{bc}]$$

after some lengthy but straightforward calculations. Finally the defining expression $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b$ yields that the curvature 2-forms of the ‘‘background‘‘ and the ‘‘full‘‘ geometry are related via

$$R_{ab} = \bar{R}_{ab} - \bar{e}^c \wedge \bar{D}(\bar{D}_b \varphi_{ac} - \bar{D}_a \varphi_{bc}).$$

When all of these preliminary results are carefully used in (8), one obtains the following expression for the ‘‘linearized‘‘ energy-momentum tensor of the cosmological Einstein theory:

$$\tau_c = -\frac{1}{2}\bar{R}^{ab} \wedge \varphi^d{}_p \bar{e}^p \wedge \bar{\star}\bar{e}_{abcd} - \frac{1}{2}[\bar{D}(\bar{D}^b \varphi^a{}_k - \bar{D}^a \varphi^b{}_k)] \wedge \bar{e}^k \wedge \bar{\star}\bar{e}_{abc} + \Lambda \varphi^b{}_p \bar{e}^p \wedge \bar{\star}\bar{e}_{cb}. \quad \text{(A2)}$$

Let us now examine the terms in τ_c (A2) individually. Using (9), the fact that

$$\bar{e}^{abp} \wedge \bar{\star}\bar{e}_{abcd} = (D-3)\bar{e}^{pa} \wedge \bar{\star}\bar{e}_{acd} = (D-3)(D-2)\bar{e}^p \wedge \bar{\star}\bar{e}_{cd},$$

one finds for the first term on the right hand side of (A2) that

$$-\frac{1}{2}\bar{R}^{ab} \wedge \varphi^d{}_p \bar{e}^p \wedge \bar{\star}\bar{e}_{abcd} = -\Lambda\left(\frac{D-3}{D-1}\right)\varphi^b{}_p \bar{e}^p \wedge \bar{\star}\bar{e}_{cb},$$

which can be added to the last term in (A2) to yield

$$-\frac{1}{2}\bar{R}^{ab} \wedge \varphi^d{}_p \bar{e}^p \wedge \bar{\star}\bar{e}_{abcd} + \Lambda \varphi^b{}_p \bar{e}^p \wedge \bar{\star}\bar{e}_{cb} = \frac{2\Lambda}{D-1}(\varphi^b{}_b \bar{\star}\bar{e}_c - \varphi^b{}_c \bar{\star}\bar{e}_b),$$

where we have also used $\bar{e}^p \wedge \bar{\star}\bar{e}_{cb} = \delta^p{}_b \bar{\star}\bar{e}_c - \delta^p{}_c \bar{\star}\bar{e}_b$.

The middle term on the right hand side of (A2) can be simplified by first noting that it can be written as

$$-\frac{1}{2}[\bar{D}(\bar{D}^b \varphi^a{}_k - \bar{D}^a \varphi^b{}_k)] \wedge \bar{e}^k \wedge \bar{\star}\bar{e}_{abc} = \frac{1}{4}(\bar{D}_p \bar{D}^a \varphi^b{}_k - \bar{D}_p \bar{D}^b \varphi^a{}_k)\bar{e}^{pk} \wedge \bar{\star}\bar{e}_{abc} - \frac{1}{4} \times (\bar{D}_k \bar{D}^a \varphi^b{}_p - \bar{D}_k \bar{D}^b \varphi^a{}_p)\bar{e}^{pk} \wedge \bar{\star}\bar{e}_{abc}$$

and this in turn can be further reduced by the fact that

$$\bar{e}^{pk} \wedge \bar{\star}\bar{e}_{abc} = (\delta^k{}_b \delta^p{}_a - \delta^k{}_a \delta^p{}_b)\bar{\star}\bar{e}_c + \text{cyclic terms in } (a, b, c).$$

Using this, one finally obtains for the middle term on the right hand side of (A2) that

$$-\frac{1}{2}[\bar{D}(\bar{D}^b \varphi^a{}_k - \bar{D}^a \varphi^b{}_k)] \wedge \bar{e}^k \wedge \bar{\star}\bar{e}_{abc} = (\bar{D}_a \bar{D}^a \varphi^b{}_b - \bar{D}_a \bar{D}^b \varphi^a{}_b)\bar{\star}\bar{e}_c + (\bar{D}_c \bar{D}^b \varphi^a{}_b - \bar{D}_c \bar{D}^a \varphi^b{}_b + \bar{D}_b \bar{D}^a \varphi^b{}_c - \bar{D}_b \bar{D}^b \varphi^a{}_c)\bar{\star}\bar{e}_a.$$

Combining all of these results finally gives

$$\tau_c = \eta_{ca}\left(-\bar{D}_p \bar{D}^b \varphi^p{}_b + \bar{D}_p \bar{D}^p \varphi^b{}_b + \frac{2\Lambda}{D-1}\varphi^p{}_p\right)\bar{\star}\bar{e}^a + \left(-\bar{D}_c \bar{D}_a \varphi^b{}_b + \bar{D}_c \bar{D}^b \varphi_{ab} - \bar{D}_b \bar{D}^b \varphi_{ac} + \bar{D}_b \bar{D}_a \varphi^b{}_c - \frac{2\Lambda}{D-1}\varphi_{ac}\right)\bar{\star}\bar{e}^a$$

for the ‘‘linearized‘‘ energy-momentum tensor of the D-dimensional cosmological Einstein theory. Hence (10) readily follows from this expression for τ_c.

APPENDIX B: THE DERIVATION OF (11)

In this appendix, we show the details of how (11) is obtained from (10). For this purpose, first note the following identities:

Since $\bar{\xi}_a$ is a Killing vector, it immediately follows from the Killing Eq. (6) that $\bar{D}_a \bar{\xi}^a = 0$. Moreover, the very definition of the Riemann tensor implies that

$$(\bar{D}_a \bar{D}_b - \bar{D}_b \bar{D}_a)\bar{\xi}_c = \bar{R}_{abcd}\bar{\xi}^d.$$

This can be used together with the key property of the Riemann tensor [This identity can easily be derived from $D\theta^a = R^a{}_b \wedge e^b = 0$ where $\theta^a \equiv \bar{D}\bar{e}^a$ denotes the torsion 2-form.].

$$\bar{R}_{[abc]d} = 0 \quad \text{and hence} \quad \bar{R}_{[abc]d}\bar{\xi}^d = 0,$$

to obtain $\bar{D}_b \bar{D}_c \bar{\xi}^a = \bar{R}^a{}_{cbd}\bar{\xi}^d$. This further simplifies by making use of (9) and leads to the useful identity that

$$\bar{D}_b \bar{D}_c \bar{\xi}^a = \frac{2\Lambda}{(D-1)(D-2)}(\delta^a{}_b \bar{\xi}_c - \eta_{bc}\bar{\xi}^a). \quad \text{(B1)}$$

Consider, for example, the first term in (10). One has

$$\begin{aligned}\bar{\xi}^a\bar{D}_c\bar{D}^b\varphi^c{}_b &= \bar{D}_c(\bar{\xi}^a\bar{D}^b\varphi^c{}_b) - \bar{D}^b(\varphi^c{}_b\bar{D}_c\bar{\xi}^a) \\ &\quad + \varphi^c{}_b(\bar{D}^b\bar{D}_c\bar{\xi}^a) \\ &= \bar{D}_c(\bar{\xi}^a\bar{D}^b\varphi^c{}_b - \varphi^{bc}\bar{D}_b\bar{\xi}^a) \\ &\quad + \frac{2\Lambda}{(D-1)(D-2)}(\varphi_c{}^a\bar{\xi}^c - \varphi^c{}_c\bar{\xi}^a).\end{aligned}$$

We have carried the $\bar{\xi}^a$ term "inside" the derivative operator in the first line and used (B1) to obtain the second line. One follows similar steps for the other terms in (10), and noting that all terms of the type $\varphi^{ab}\bar{\xi}_b$ and $\varphi^b{}_b\bar{\xi}^a$ cancel out separately along the way, the final expression for (11) is found.

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