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Energy in generic higher curvature gravity theories

S. Deser* and Bayram Tekin†
Department of Physics, Brandeis University, Waltham, Massachusetts 02454

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We define and compute the energy of higher curvature gravity theories in arbitrary dimensions. Generically, these theories admit constant curvature vacua (even in the absence of an explicit cosmological constant), and asymptotically constant curvature solutions with nontrivial energy properties. For concreteness, we study quadratic curvature models in detail. Among them, the one whose action is the square of the traceless Ricci tensor always has zero energy, unlike conformal (Weyl) gravity. We also study the string-inspired Einstein-Gauss-Bonnet model and show that both its flat and anti–de Sitter vacua are stable.

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I. INTRODUCTION

Recently, de Sitter (dS) and anti–de Sitter (AdS) spaces have received renewed interest both in string theory [AdS -conformal field theory (CFT) correspondence] and in cosmology where a positive cosmological constant may have been observed. This motivates a detailed study of energy about these vacua, for systems that also involve higher curvature terms, such as naturally arise in string theory and other quantum gravity models. In this paper, we define and compute the global charges (especially energy) of asymptotically constant (including zero) curvature space-times for generic gravitational models.

In a recent Letter [1], which summarized some of the present work, we defined the global charges primarily in four dimensional quadratic theories. In this paper we extend that discussion in several directions: We first present a reformulation of the original definition [2] of conserved charges in cosmological Einstein theory; then we derive the generic form of the energy for quadratic gravity theories in D dimensions and specifically study the ghost-free low energy string-inspired model: Gauss-Bonnet (GB) plus Einstein terms. We also briefly indicate how higher curvature models can be similarly treated.

We will demonstrate that, among purely quadratic theories, the one whose Lagrangian is the square of the traceless Ricci tensor has zero energy for all D about its asymptotically flat or asymptotically constant curvature vacua, unlike, for example, conformal (Weyl) gravity in $D=4$.

A definition of gauge invariant conserved (global) charges in a diffeomorphism-invariant theory rests on the "Gauss law" and the presence of asymptotic Killing symmetries. More explicitly, in *any* diffeomorphism-invariant gravity theory, a vacuum satisfying the classical equations of motion is chosen as the background relative to which excitations and any background gauge-invariant properties (such as energy) are defined. Two important model-independent features of these charges are: First, the vacuum itself has zero charge; secondly, they are expressible as surface integrals. As we shall show below, a generic formulation (independent of the gravity model considered) is rather simple and straightforward, although its applications to specific gravity models require care, in choosing correct vacua, with global symmetries and in converting "volume" to "surface" integrals. Historically, the first application of this procedure was in Einstein's gravity for flat backgrounds with its Poincaré symmetries ["Arnowitt-Deser-Misner (ADM) mass" [3]]. The second step was to the (A)dS vacua of cosmological Einstein theory ["Abbott-Deser (AD) mass" [2]].

The outline of the paper is as follows: In Sec. II, we reexamine the AD [2] Killing charge for the cosmological Einstein theory and the energies of its Schwarzschild-(A)dS (collectively "SdS") solutions. Section III is devoted to the derivation and computation of the Killing charges in generic quadratic theories (with or without Einstein terms) as well as their various limits, particularly in Einstein-GB models. In Sec. IV, we discuss the purely quadratic zero energy theory constructed from the traceless Ricci tensor. Section V includes our conclusions and some open questions. The Appendix collects some formulas useful for linearization properties of quadratic curvature terms about (A)dS backgrounds.

II. REFORMULATION OF AD ENERGY

In this section, we reformulate the AD construction [2] and obtain new and perhaps more transparent surface integrals for energy in cosmological Einstein theory. One of the reasons for revisiting the AD formulation is, as will become clear, that in the higher curvature models we shall study in detail, the only non-vanishing parts of energy, for asymptotically SdS spaces come precisely from AD integrals, but with essential contributions from the higher terms.

First, let us recapitulate [1] how conserved charges arise in a generic gravity theory coupled to a covariantly conserved bounded matter source $\tau_{\mu\nu}$,

$$\Phi_{\mu\nu}(g,R,\nabla R,R^2,\dots)=\kappa\tau_{\mu\nu}, \tag{1}$$

where $\Phi_{\mu\nu}$ is the "Einstein tensor" of a local, invariant, but otherwise arbitrary, gravity action and κ is an effective coupling constant. Now decompose the metric into the sum of a background $\bar{g}_{\mu\nu}$ [which solves Eq. (1) for $\tau_{\mu\nu}=0$] plus a (not necessarily small) deviation $h_{\mu\nu}$, that vanishes sufficiently rapidly at infinity,

*Email address: deser@brandeis.edu
†Email address: tekin@brandeis.edu

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}. \tag{2}$$

Separating the field equations (1) into a part linear in $h_{\mu\nu}$ plus all the non-linear ones that constitute the total source $T_{\mu\nu}$, including the matter source $\tau_{\mu\nu}$, one obtains

$$\mathcal{O}(\bar{g})_{\mu\nu\alpha\beta}h^{\alpha\beta}=\kappa T_{\mu\nu}, \tag{3}$$

as $\Phi_{\mu\nu}(\bar{g},\bar{R},\bar{\nabla}\bar{R},\bar{R}^2\ldots)=0$, by assumption; the Hermitian operator $\mathcal{O}(\bar{g})$ depends only on the background metric (that also moves all indices and defines the covariant derivatives $\bar{\nabla}_\mu$). It is clear that this operator inherits both background Bianchi identity and background gauge invariance, namely $\bar{\nabla}_\mu\mathcal{O}(\bar{g})^{\mu\nu\alpha\beta}=\mathcal{O}(\bar{g})^{\mu\nu\alpha\beta}\bar{\nabla}_\alpha=0$, from (the Bianchi identities of) the full theory. As a consequence of these invariances, it is guaranteed that if the background $\bar{g}_{\mu\nu}$ admits a set of Killing vectors $\bar{\xi}_\mu^{(a)}$,

$$\bar{\nabla}_\mu\bar{\xi}_\nu^{(a)}+\bar{\nabla}_\nu\bar{\xi}_\mu^{(a)}=0, \tag{4}$$

then they can be used to construct the following (ordinarily) conserved vector density current,

$$\bar{\nabla}_\mu(\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu^a)\equiv\partial_\mu(\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu^a)=0. \tag{5}$$

Therefore, the conserved Killing charges are expressed as

$$Q^\mu(\bar{\xi}^a)=\int_{\mathcal{M}}d^{D-1}x\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu^a=\int_\Sigma dS_i\mathcal{F}^{\mu i}. \tag{6}$$

Here $\mathcal{M}$ is a spatial $(D-1)$ hypersurface and Σ is its $(D-2)$ dimensional boundary; as will follow from Eq. (13), $\mathcal{F}^{\mu i}$ is an antisymmetric tensor obtained from $\mathcal{O}(\bar{g})$, whose explicit form, of course, depends on the theory.

Let us first apply the above procedure to cosmological Einstein theory to rejoin [2]. Our conventions are: signature $(-,+,+,\ldots+)$, $[\nabla_\mu,\nabla_\nu]V_\lambda=R_{\mu\nu\lambda}{}^\sigma V_\sigma$, $R_{\mu\nu}\equiv R_{\mu\lambda\nu}{}^\lambda$. The Einstein equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=0, \tag{7}$$

are solved by the constant curvature vacuum $\bar{g}_{\mu\nu}$, whose Riemann, Ricci and scalar curvature are

$$\bar{R}_{\mu\lambda\nu\beta}=\frac{2}{(D-2)(D-1)}\Lambda(\bar{g}_{\mu\nu}\bar{g}_{\lambda\beta}-\bar{g}_{\mu\beta}\bar{g}_{\lambda\nu}),$$

$$\bar{R}_{\mu\nu}=\frac{2}{D-2}\Lambda\bar{g}_{\mu\nu},\qquad \bar{R}=\frac{2D\Lambda}{D-2}. \tag{8}$$

Linearization of Eq. (7) about this background yields

$$\mathcal{G}^L_{\mu\nu}\equiv R^L_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R^L-\frac{2}{D-2}\Lambda h_{\mu\nu}\equiv\kappa T_{\mu\nu}, \tag{9}$$

where $R^L=(g^{\mu\nu}R_{\mu\nu})_L$ and the linear part of the Ricci tensor reads

$$R^L_{\mu\nu}\equiv R_{\mu\nu}-\bar{R}_{\mu\nu}=\frac{1}{2}(-\bar{\Box}h_{\mu\nu}-\bar{\nabla}_\mu\bar{\nabla}_\nu h+\bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\sigma\mu}$$
$$+\bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\sigma\nu}), \tag{10}$$

with $h=\bar{g}^{\mu\nu}h_{\mu\nu}$ and $\bar{\Box}=\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu$. The energy momentum-tensor (9) is background covariantly constant ($\bar{\nabla}_\mu T^{\mu\nu}=0$), as can be checked explicitly.

This procedure led [2] to the following energy expression:

$$E(\bar{\xi})=\frac{1}{8\pi G}\int dS_i\sqrt{-\bar{g}}\{\bar{\xi}_\nu\bar{\nabla}_\beta K^{0i\nu\beta}-K^{0j\nu i}\bar{\nabla}_j\bar{\xi}_\nu\}. \tag{11}$$

The superpotential $K^{\mu\alpha\nu\beta}$ is defined by

$$K^{\mu\nu\alpha\beta}\equiv\frac{1}{2}[\bar{g}^{\mu\beta}H^{\nu\alpha}+\bar{g}^{\nu\alpha}H^{\mu\beta}-\bar{g}^{\mu\nu}H^{\alpha\beta}-\bar{g}^{\alpha\beta}H^{\mu\nu}],$$

$$H^{\mu\nu}=h^{\mu\nu}-\frac{1}{2}\bar{g}^{\mu\nu}h. \tag{12}$$

It has the symmetries of the Riemann tensor. In converting the volume to surface integrals, we follow a somewhat different route, which will be convenient in the higher curvature cases. Using Eqs. (9), (10), straightforward rearrangements of terms, and the aforementioned antisymmetry, we can move the covariant derivatives to yield

$$2\bar{\xi}_\nu\mathcal{G}_L^{\mu\nu}=2\bar{\xi}_\nu R_L^{\mu\nu}-\bar{\xi}_\nu\bar{g}^{\mu\nu}R^L-\frac{4\Lambda}{D-2}\bar{\xi}_\nu h^{\mu\nu}$$
$$=\bar{\xi}_\nu\{-\bar{\Box}h^{\mu\nu}-\bar{\nabla}^\mu\bar{\nabla}^\nu h+\bar{\nabla}_\sigma\bar{\nabla}^\nu h^{\sigma\mu}+\bar{\nabla}_\sigma\bar{\nabla}^\mu h^{\sigma\nu}\}$$
$$-\bar{\xi}^\mu\left\{-\bar{\Box}h+\bar{\nabla}_\sigma\bar{\nabla}_\nu h^{\sigma\nu}-\frac{2\Lambda}{D-2}h\right\}-\frac{4\Lambda}{D-2}\bar{\xi}_\nu h^{\mu\nu}$$
$$=\bar{\nabla}_\rho\{\bar{\xi}_\nu\bar{\nabla}^\mu h^{\rho\nu}-\bar{\xi}_\nu\bar{\nabla}^\rho h^{\mu\nu}+\bar{\xi}^\mu\bar{\nabla}^\rho h-\bar{\xi}^\rho\bar{\nabla}^\mu h$$
$$+h^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\nu-h^{\rho\nu}\bar{\nabla}^\mu\bar{\xi}_\nu+\bar{\xi}^\rho\bar{\nabla}_\nu h^{\mu\nu}-\bar{\xi}^\mu\bar{\nabla}_\nu h^{\rho\nu}$$
$$+h\bar{\nabla}^\mu\bar{\xi}^\rho\}. \tag{13}$$

Since the charge densities are surface terms, the Killing charges become those of [2]:

$$Q^\mu(\bar{\xi})=\frac{1}{4\Omega_{D-2}G_D}\int_\Sigma dS_i\ \{\bar{\xi}_\nu\bar{\nabla}^\mu h^{i\nu}-\bar{\xi}_\nu\bar{\nabla}^i h^{\mu\nu}+\bar{\xi}^\mu\bar{\nabla}^i h$$
$$-\bar{\xi}^i\bar{\nabla}^\mu h+h^{\mu\nu}\bar{\nabla}^i\bar{\xi}_\nu-h^{i\nu}\bar{\nabla}^\mu\bar{\xi}_\nu+\bar{\xi}^i\bar{\nabla}_\nu h^{\mu\nu}-\bar{\xi}^\mu\bar{\nabla}_\nu h^{i\nu}$$
$$+h\bar{\nabla}^\mu\bar{\xi}^i\}, \tag{14}$$

where i ranges over $(1,2,\ldots,D-2)$; the charge is normalized by dividing by the (D-dimensional) Newton's constant and solid angle. Before we perform the explicit computation of the energy Q^0 in specific coordinates for asymptotically (A)dS spaces, let us check that it is in fact background gauge-invariant. Under an infinitesimal diffeomorphism, generated by a vector ζ_μ, the deviation part of the metric transforms as

$$\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu. \tag{15}$$

To show that $T_{\mu\nu}\bar{\xi}^\nu$ is invariant, first note that R_L is

$$\delta_\zeta R_L = \bar{g}^{\mu\nu}\delta_\zeta R^L_{\mu\nu} - \frac{2}{D-2}\Lambda \bar{g}^{\mu\nu}\delta_\zeta h_{\mu\nu} = 0. \tag{16}$$

This leads to $\delta_\zeta \mathcal{G}^L_{\mu\nu} = [2/(D-2)]\Lambda\,\delta_\zeta h_{\mu\nu}$ and eventually to $\delta_\zeta Q^\mu = 0$: the Killing charge is indeed background gauge-invariant. Another test of Eq. (14) is that, in the limit of an asymptotically flat background, we should obtain the ADM charge. In that case, we may write the timelike Killing vector as $\xi_\mu = (1,\mathbf{0})$. The time component of Eq. (14) reduces to the desired result:

$$Q^0 = M_{ADM} = \frac{1}{4\Omega_{D-2}G_D}\int_\Sigma dS_i\,\{\partial_j h^{ij} - \partial^i h_{jj}\} \tag{17}$$

in terms of Cartesian coordinates.

Having established the energy formula for asymptotically (A)dS spaces, we can now evaluate the energy of SdS solutions. First, we must recall that the existence of a cosmological horizon is an important difference between dS and AdS cases. In the former, the background Killing vector stays time-like only within the cosmological horizon. (We will not go into the complications for physics of this horizon, since it is a well-known and ongoing problem. In [2], it was simply assumed that interesting system should be describable within the horizon. For related ideas see [4].) For small black holes, whose own event horizons lie well inside the cosmological one, Eq. (14) provides a reasonable approximation.

In static coordinates, the line element of D-dimensional SdS reads

$$ds^2 = -\left\{1 - \left(\frac{r_0}{r}\right)^{D-3} - \frac{r^2}{l^2}\right\}dt^2 + \left\{1 - \left(\frac{r_0}{r}\right)^{D-3} - \frac{r^2}{l^2}\right\}^{-1} dr^2 + r^2 d\Omega^2_{D-2}, \tag{18}$$

where $l^2 \equiv (D-2)(D-1)/2\Lambda$. The background ($r_0=0$) Killing vector is $\bar{\xi}^\mu = (-1,\mathbf{0})$, which is time-like everywhere for AdS ($l^2<0$) but remains time-like for dS ($l^2>0$) only inside the cosmological horizon: $\bar{g}_{\mu\nu}\bar{\xi}^\mu\bar{\xi}^\nu = -(1-r^2/l^2)$.

Let us concentrate on $D=4$ first and calculate the surface integral (14) not at $r=\infty$, but at some finite distance r from the origin; this will *not* be gauge-invariant, since energy is to be measured only at infinity. Nevertheless, for dS space (which has a horizon that keeps us from going smoothly to infinity), let us first keep r finite as an intermediate step. The integral becomes

$$E(r) = \frac{r_0}{2G}\frac{\left(1-\frac{r^2}{l^2}\right)}{\left(1-\frac{r_0}{r}-\frac{r^2}{l^2}\right)}. \tag{19}$$

For AdS, $E(r\to\infty) = r_0/2G \equiv M$, as expected. On the other hand, for dS $E(r=l)=0$. This is, however, misleading since in dS we should really only consider small r_0 objects, which do not change the location of the background horizon. [Indeed, if we naively include the effect of a large r_0 as changing the horizon to $1-r_0/r-r^2/l^2=0$, then $E(r)$ itself diverges.] But, we derived the energy formula using asymptotic Killing vectors, so the only way to make sense of the above result for asymptotically dS spaces is to consider the small r_0 limit, which then gives $E=M$ [2]. In the limit of a vanishing cosmological constant, $l\to\infty$; the ADM energy is of course recovered as $r\to\infty$.

The above argument easily generalizes to D-dimensions, where one obtains

$$E = \frac{D-2}{4G_D}r_0^{D-3}. \tag{20}$$

Here r_0 can be arbitrarily large in the AdS case but must be small in dS.

Finally, let us note that analogous computations can also be carried out in $D=3$; the proper solution is

$$ds^2 = -\left(1-r_0-\frac{r^2}{l^2}\right)dt^2 + \left(1-r_0-\frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\phi^2 \tag{21}$$

for which the energy is $E=r_0/2G$ again but, now, r_0 is a dimensionless constant and $[G]=[M^{-1}]$, in agreement with the original results [5].

III. STRING-INSPIRED GRAVITY

In flat backgrounds, the ghost freedom of low energy string theory requires the quadratic corrections to Einstein's gravity to be of the GB form [6], an argument that should carry over to the AdS backgrounds. Below we construct and compute the energy of various asymptotically (A)dS spaces that solve generic Einstein plus quadratic gravity theories, particularly the Einstein-GB model.

At quadratic order, the generic action is[1]

$$I = \int d^D x\sqrt{-g}\left\{\frac{R}{\kappa} + \alpha R^2 + \beta R^2_{\mu\nu} + \gamma(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2)\right\}. \tag{22}$$

In $D=4$, the GB part (γ terms) is a surface integral and plays no role in the equations of motion. In $D>4$, on the

[1]We will later add an explicit cosmological constant term in the discussion. Note also that the normalizations of α,β differ from those of [1].

contrary, GB is the only viable term, since non-zero α, β produce ghosts [7]. Here $\kappa=2\Omega_{D-2}G_D$, where G_D is the D-dimensional Newton's constant.

The equations of motion that follow from Eq. (22) are

$$\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+2\alpha R\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)+(2\alpha+\beta)\times(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R+2\gamma\Big\{RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu^{\sigma\rho\tau}-2R_{\mu\sigma}R^\sigma_\nu-\frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\rho\sigma}-4R^2_{\sigma\rho}+R^2)\Big\}+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+2\beta\left(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho}=\tau_{\mu\nu}. \tag{23}$$

In the absence of matter, flat space is *a* solution of these equations. But so is (A)dS with cosmological constant Λ, which in our conventions is (see also [8])

$$-\frac{1}{2\Lambda\kappa}=\frac{(D-4)}{(D-2)^2}(D\alpha+\beta)+\frac{\gamma(D-4)(D-3)}{(D-2)(D-1)},\quad \Lambda\neq 0. \tag{24}$$

Several comments are in order here. In the string-inspired Einstein-GB model ($\alpha=\beta=0$ and $\gamma>0$), *only* AdS background ($\Lambda<0$) is allowed (the Einstein constant κ is positive with our conventions). String theory is known to prefer AdS to dS (see for example the no-go theorem [9]) we can now see why this is so in the uncompactified theory. Another interesting limit is the "traceless" theory ($D\alpha=-\beta$), which, in the absence of a γ term, does not allow constant curvature spaces unless the Einstein term is also dropped. For $D=4$, the γ term drops out, and the pure quadratic theory allows (A)dS solutions with arbitrary Λ. For $D>4$, relation (24) leaves a 2-parameter set (say α,β) of allowed solutions.

Following the procedure outlined in the previous section and using the formulas in the Appendix, we expand the field equations to first order in $h_{\mu\nu}$ and define the total energy momentum tensor as

$$T_{\mu\nu}(h)\equiv T_{\mu\nu}(\bar g)+\mathcal{G}^L_{\mu\nu}\left\{\frac{1}{\kappa}+\frac{4\Lambda D\alpha}{D-2}+\frac{4\Lambda\beta}{D-1}+\frac{4\Lambda\gamma(D-4)(D-3)}{(D-2)(D-1)}\right\}+(2\alpha+\beta)\left(\bar g_{\mu\nu}\bar\Box-\bar\nabla_\mu\bar\nabla_\nu+\frac{2\Lambda}{D-2}g_{\mu\nu}\right)R_L+\beta\left(\bar\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{D-1}\bar g_{\mu\nu}R_L\right)-2\Lambda^2h_{\mu\nu}\left\{\frac{1}{2\Lambda\kappa}+\frac{(D-4)}{(D-2)^2}(D\alpha+\beta)+\frac{\gamma(D-4)(D-3)}{(D-2)(D-1)}\right\}. \tag{25}$$

Using Eq. (24) one has $T_{\mu\nu}(\bar g)=0$ and the last term also vanishes, yielding

$$T_{\mu\nu}=\mathcal{G}^L_{\mu\nu}\left\{-\frac{1}{\kappa}+\frac{4\Lambda D}{(D-2)^2}\left(2\alpha+\frac{\beta}{D-1}\right)\right\}+(2\alpha+\beta)\times\left(\bar g_{\mu\nu}\bar\Box-\bar\nabla_\mu\bar\nabla_\nu+\frac{2\Lambda}{D-2}g_{\mu\nu}\right)R_L+\beta\left(\bar\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{D-1}\bar g_{\mu\nu}R_L\right). \tag{26}$$

This is a background conserved tensor ($\bar\nabla^\mu T_{\mu\nu}=0$) as can be checked explicitly with the help of the expressions

$$\bar\nabla^\mu\left(\bar g_{\mu\nu}\bar\Box-\bar\nabla_\mu\bar\nabla_\nu+\frac{2\Lambda}{D-2}g_{\mu\nu}\right)R_L=0$$

$$\bar\nabla^\mu\left(\bar\Box\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{D-1}\bar g_{\mu\nu}\right)R_L=0. \tag{27}$$

An important aspect of Eq. (26) is the sign change of the $1/\kappa$ term relative to Einstein theory, due to the GB contributions as already noticed in [10]. Hence in the Einstein-GB limit, we have $T_{\mu\nu}=-\mathcal{G}^L_{\mu\nu}/\kappa$, with overall sign exactly opposite[2] to that of the cosmological Einstein theory (9). But, as we shall see below, this does not mean that E is negative there.

There remains now to obtain a Killing energy expression from Eq. (26), namely, to write $\bar\xi_\nu T^{\mu\nu}$ as a surface integral. The first term is the usual AD piece (14), which we have already dealt with in the previous section. The middle term with the coefficient $2\alpha+\beta$, is easy to handle. The relatively cumbersome last term can be written as a surface plus extra terms:

$$\bar\xi_\nu\bar\Box\mathcal{G}^L_{\mu\nu}=\bar\nabla_\alpha\{\bar\xi_\nu\bar\nabla^\alpha\mathcal{G}^{\mu\nu}_L-\bar\xi_\nu\bar\nabla^\mu\mathcal{G}^{\alpha\nu}_L-\mathcal{G}^{\mu\nu}_L\bar\nabla^\alpha\bar\xi_\nu+\mathcal{G}^{\alpha\nu}_L\bar\nabla^\mu\bar\xi_\nu\}+\mathcal{G}^{\mu\nu}_L\bar\Box\bar\xi_\nu+\bar\xi_\nu\bar\nabla_\alpha\bar\nabla^\mu\mathcal{G}^{\alpha\nu}-\mathcal{G}^{\alpha\nu}_L\bar\nabla_\alpha\bar\nabla^\mu\bar\xi_\nu. \tag{28}$$

Using the definition of the Killing vector, and its trace property,

$$\bar\nabla_\alpha\bar\nabla_\beta\bar\xi_\nu=\bar R^\mu_{\nu\beta\alpha}\bar\xi_\mu=\frac{2\Lambda}{(D-2)(D-1)}(\bar g_{\nu\alpha}\bar\xi_\beta-\bar g_{\alpha\beta}\bar\xi_\nu),$$

$$\bar\Box\bar\xi_\mu=-\frac{2\Lambda}{D-2}\bar\xi_\mu, \tag{29}$$

along with the identity

$$\bar\xi_\nu\bar\nabla_\alpha\bar\nabla^\mu\mathcal{G}^{\alpha\nu}_L=\frac{2\Lambda D}{(D-2)(D-1)}\bar\xi_\nu\mathcal{G}^{\mu\nu}_L+\frac{\Lambda}{D-1}\xi^\mu R_L, \tag{30}$$

[2]This overall sign change is also shared by the model's small oscillations about the AdS vacuum.

one can show that $\bar{\xi}_\nu \bar{\Box} \mathcal{G}^L_{\mu\nu}$ can indeed be written as a surface term. Collecting everything, the final form of the conserved charges for the generic quadratic theory reads

$$Q^\mu(\bar{\xi}) = \left\{ -\frac{1}{\kappa} + \frac{8\Lambda}{(D-2)^2}(D\alpha+\beta) \right\} \times \int d^{D-1}x \sqrt{-\bar{g}}\, \bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} + (2\alpha+\beta) \times \int dS_i \sqrt{-\bar{g}} \{ \bar{\xi}^\mu \bar{\nabla}^i R_L + R_L \bar{\nabla}^\mu \bar{\xi}^i - \bar{\xi}^i \bar{\nabla}^\mu R_L \} + \beta \times \int dS_i \sqrt{-\bar{g}} \{ \bar{\xi}_\nu \bar{\nabla}^i \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{i\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^i \bar{\xi}_\nu + \mathcal{G}_L^{i\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \}. \tag{31}$$

For brevity we have left the AD part as a volume integral whose surface form we know is given by Eq. (14); note that γ does not appear explicitly since it has been traded for Λ through the relation (24).

In the above analysis, there was no bare cosmological term in the action. Clearly, this need not be the case: we can add one, say $2\int d^D x \sqrt{-g}\Lambda_0/\kappa$. The Λ_0 contributes to the overall *effective* cosmological constant Λ, which now is given by

$$\Lambda = -\frac{1}{4f(\alpha,\beta,\gamma)\kappa} \times \{1 \pm \sqrt{1+8\kappa f(\alpha,\beta,\gamma)\Lambda_0}\} \tag{32}$$

$$f(\alpha,\beta,\gamma) \equiv \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \frac{\gamma(D-4)(D-3)}{(D-2)(D-1)}.$$

If $f>0$, as in Einstein-GB theory, the effective cosmological constant Λ is smaller than the "bare" one Λ_0: thus stringy corrections (at quadratic order) reduce the value of the bare cosmological constant appearing in the Lagrangian. Given that Λ_0 is arbitrary, there is a bound $(8\kappa\Lambda_0 f \geqslant -1)$ on these corrections since the effective Λ becomes imaginary otherwise.

Now let us compute the energy of an asymptotically SdS geometry that might be a solution to our generic model. Should such a solution exist, we only require its asymptotic behavior to be

$$h_{00} \approx + \left(\frac{r_0}{r}\right)^{D-3}, \quad h^{rr} \approx + \left(\frac{r_0}{r}\right)^{D-3} + O(r_0^2). \tag{33}$$

It is easy to see that for asymptotically SdS spaces the second and the third lines of Eq. (31) do not contribute, since for any Einstein space, to linear order

$$R^L_{\mu\nu} = \frac{2\Lambda}{D-2} h_{\mu\nu}, \tag{34}$$

which in turn yields $R_L = \bar{g}^{\mu\nu} R^L_{\mu\nu} - [2\Lambda/(D-2)]h = 0$ and thus $\mathcal{G}^L_{\mu\nu} = 0$ in the asymptotic region. Therefore the total energy of the full (α,β,γ) system, for geometries that are asymptotically SdS, is given only by the first term in Eq. (31),

$$E_D = \left\{ -1 + \frac{8\Lambda\kappa}{(D-2)^2}(D\alpha+\beta) \right\} \frac{(D-2)}{4G} r_0^{D-3}, \quad D>4, \tag{35}$$

where γ is implicitly assumed not to vanish. (Note again the sign change of the "Einstein contribution" as explained before.) For $D=4$, we computed E in [1]; equivalently from Eq. (25), it reads[3] (for models with an explicit Λ)

$$E_4 = \{1 + 2\Lambda\kappa(4\alpha+\beta)\} \frac{r_0}{2G}. \tag{36}$$

From Eq. (35), the asymptotically SdS solution seemingly has negative energy, in the Einstein-GB model:

$$E = -\frac{(D-2)}{4G} r_0^{D-3}. \tag{37}$$

While this is of course correct in terms of the usual SdS signs, one must be more careful about the external solutions in Einstein-GB theory. Their exact form is [10]

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 + r^2 d\Omega_{D-2} \tag{38}$$

$$-g_{00} = g_{rr}^{-1} = 1 + \frac{r^2}{4\kappa\gamma(D-3)(D-4)} \times \left\{ 1 \pm \left\{ 1 + 8\gamma(D-3)(D-4)\frac{r_0^{D-3}}{r^{D-1}} \right\}^{1/2} \right\}. \tag{39}$$

Note that there is a branching here, with qualitatively different asymptotics: Schwarzschild and Schwarzschild-AdS,

$$-g_{00} = 1 - \left(\frac{r_0}{r}\right)^{D-3},$$
$$-g_{00} = 1 + \left(\frac{r_0}{r}\right)^{D-3} + \frac{r^2}{\kappa\gamma(D-3)(D-4)}. \tag{40}$$

[Here we have restored γ, using $\kappa\gamma(D-3)(D-4) = -l^2$.] The first solution has the usual positive (for positive r_0 of course) ADM energy $E = +(D-2)r_0^{D-3}/4G$, since the GB term does not contribute when expanded around flat space. On the other hand, as noted in [10] the second solution which is asymptotically SdS, has the wrong sign for the "mass term." But, to actually compute the energy here, one

[3] In $D=3$, the GB density vanishes identically and the energy expression has the same form of the $D=4$ model, with the difference that r_0 comes from the metric (21).

needs our energy expression (31), and not simply the AD formula which is valid only for cosmological Einstein theory. Now from Eq. (40), we have

$$h_{00} \approx -\left(\frac{r_0}{r}\right)^{D-3}, \quad h^{rr} \approx -\left(\frac{r_0}{r}\right)^{D-3} + O(r_0^2), \tag{41}$$

whose sign is opposite to that of the usual SdS. This sign just compensates the flipped sign in the energy definition, so the energy (35) reads $E=(D-2)r_0^{D-3}/4G$ and the AdS branch, just like the flat branch, has positive energy, after the GB effects are taken into account also in the energy definition. Thus, for every Einstein-GB external solution, energy is positive and AdS vacuum is stable.[4]

IV. ZERO ENERGY MODELS

In $D=4$, every quadratic curvature theory, i.e. any (α,β) combination, is scale invariant. These models were studied in [12] in terms of the slightly different parametrization

$$S=\int d^4x\sqrt{-g}\{aC_{\mu\nu\sigma\rho}\ C^{\mu\nu\sigma\rho}+bR^2\} \tag{42}$$

where $C_{\mu\nu\sigma\rho}$ is the Weyl tensor. Using the equivalent of the ADM energy for the asymptotically flat solutions, it was shown that this energy vanished for all of them. As discussed in [1], with our definition of energy, this statement is correct, but simply reflects (at Einsteinian level) the Newtonian impossibility of having asymptotically vanishing solutions of $\nabla^4\phi=\rho$. This property of higher derivative gravity is well understood [13]. It has deeper consequences such as violations of the equivalence principle: massive sources here have no gravitational mass. Violations of the equivalence principle are not unheard of and occur already at the simple level of scalar-tensor gravity. In the asymptotically (A)dS branch, however, energy no longer vanishes: Even pure conformally invariant Weyl theory has finite energy.

Interestingly, there is one purely quadratic theory which *does* have vanishing energy in *all* dimensions, for asymptotically flat or (A)dS vacua. It has action $\int d^Dx\sqrt{-g}\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$, where $\tilde{R}_{\mu\nu}\equiv R_{\mu\nu}-(R/D)g_{\mu\nu}$. This vanishing is obvious from Eq. (35), dropping the Einstein contribution: E is then proportional to $(D\alpha+\beta)$. In addition to its zero-energy flat vacuum, the (A)dS branch is infinitely degenerate, having a 1-dimensional moduli space denoted by the Schwarzschild parameter r_0. For example, creating larger and larger black holes costs nothing in this theory. Of course, once an Einstein term is present, the energy is no longer zero.

V. CONCLUSIONS

We have defined the energy of generic Einstein plus cosmological term plus quadratic gravity theories as well as pure quadratic models in all D, for both asymptotically flat and (A)dS spaces. For flat backgrounds, the higher derivative terms do not change the form of the energy expressions. On the other hand, for asymptotically (A)dS backgrounds (which are generically solutions to these equations, even in the absence of an explicit cosmological constant), the energy expressions (31) essentially reduce to the AD formula [up to higher order corrections that vanish for space-times that asymptotically approach (A)dS at least as fast as SdS spaces].

Among quadratic theories, we have studied the string-inspired Einstein-GB model in more detail. Just like the others, this one, in the absence of an explicit cosmological constant, has both flat and AdS vacua, the latter with specific (negative) cosmological constants determined by the Newton's constant and the GB coefficient, the latter sign being fixed from the string expansion to be positive. The explicit spherically symmetric black hole solutions in this theory consist of two branches [10]: asymptotically Schwarzschild spaces with a positive mass parameter or asymptotically Schwarzschild AdS spaces with a *negative* one. The asymptotically Schwarzschild branch has the usual positive ADM energy. Using the compensation of two minus signs in the solution and in the correct energy definition, we noted that the AdS branch has likewise positive energy and that the AdS vacuum was a stable zero energy state.

Amusingly, we instead identified a unique, purely quadratic theory with zero energy for all constant (or zero) curvature backgrounds. That, one such model must exist, is already clear from the fact that each term in

$$I=\int d^Dx\sqrt{-g}\{\alpha R^2+\beta R_{\mu\nu}^2+\gamma(R_{\mu\nu\rho\sigma}^2-4R_{\mu\nu}^2+R^2)\}, \tag{43}$$

contributes linearly to E. The condition that (A)dS be a solution, with arbitrary cosmological constant Λ, is

$$(D-4)\left\{\frac{1}{(D-2)}(D\alpha+\beta)+\frac{\gamma(D-3)}{(D-1)}\right\}=0. \tag{44}$$

In all D, the zero energy models have $(D\alpha+\beta)=0=\gamma$. While we have not yet understood what this result means physically, we can at least argue in favor of its plausibility. First, note that this model is the only one that stays special in all D, unlike either Weyl gravity, good only in $D=4$ or R^2, scale invariant also only in $D=4$. A second argument is that this is the only quadratic theory that cannot be reformulated as Einstein plus matter [11], making it hard to expect any of the others to have no energy.

In this paper, we have only looked at constant curvature vacua, but there may exist more general vacua with some additional structure. One example may be Weyl gravity, for which the most general spherically symmetric solution is [14,15]

$$-g_{00}=\frac{1}{g_{rr}}=1-3ab-\frac{(2-3ab)b}{r}+ar-\frac{r^2}{l^2}; \tag{45}$$

[4]In [10], it was erroneously concluded that E_D was negative for the AdS branch, despite having obtained both the correct (negative) sign of $T^{\mu\nu}$ and of course the correct solution (39).

a,b,l are integration constants. Birkhoff's theorem is valid and this is the unique external solution. One choice of background might be to set $b=0$. This space is only asymptotically (A)dS, since for it, $R=-6a/r+12/l^2$. Our earlier remarks on the loss of "visibility" of matter source contributions to E in higher derivative theories might lead one to expect the *ar* term to carry this information. However this is not the case: the $b=0$ geometry is a solution everywhere.

The framework for energy definition presented here can clearly be applied to models with generic higher powers of curvature [1]. For any such theory that supports constant curvature vacua—and all but monomials in scalar curvature do so—it is just a matter of turning the crank to obtain its energy.

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APPENDIX

Here we list some useful linearization expressions about (A)dS for pure quadratic terms, using the conventions of Sec. II; barred quantities refer to the background:

$$\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma})=\frac{2\Lambda}{D-1}R^L_{\mu\nu}+\frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}R_L+\frac{4\Lambda^2}{(D-2)^2(D-1)}h_{\mu\nu}$$

$$\delta(R_{\mu\rho\sigma\alpha}R_{\nu}{}^{\rho\sigma\alpha})=\frac{8\Lambda}{(D-1)(D-2)}R^L_{\mu\nu}-\frac{8\Lambda^2}{(D-2)^2(D-1)}h_{\mu\nu}$$

$$\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha})=\frac{8\Lambda}{(D-1)(D-2)}R_L$$

$$\delta(RR_{\mu\nu})=\frac{2D\Lambda}{D-2}R^L_{\mu\nu}+\frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}R_L$$

$$\delta(R^{\sigma}_{\mu}R_{\nu\sigma})=\frac{4\Lambda}{D-2}R^L_{\mu\nu}-\frac{4\Lambda^2}{(D-2)^2}h_{\mu\nu}$$

$$\delta(R^2_{\mu\nu})=\frac{4\Lambda}{D-2}R_L$$

$$\delta(R^2_{\tau\lambda\rho\sigma}-4R^2_{\sigma\rho}+R^2)=\frac{4\Lambda(D-3)}{D-1}R_L$$

$$R^L_{\mu\sigma\nu\rho}\bar{g}^{\sigma\rho}=R^L_{\mu\nu}-\frac{2\Lambda}{(D-1)(D-2)}\times(h_{\mu\nu}-\bar{g}_{\mu\nu}h).$$

Finally, we compute the GB density of a cosmological space:

$$\bar{R}^2_{\tau\lambda\rho\sigma}-4\bar{R}^2_{\sigma\rho}+\bar{R}^2=\frac{4D\Lambda^2(D-3)}{(D-2)(D-1)}.$$

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